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SOME RULES OF SPECTRAL STRUCTURE*

By Otto Laporte and William F. Meggers

During the past year the important structural features of several complex spectra of the first and second long periods have been recognized, and the regularities which have been found have essentially confirmed Sommerfeld's and Landé's rules for line intensities and level intervals of the multiplet groups. But for the energy diagram of the individual atom, for the arrangement of the spectral terms relative to each other, and for the connection of these data with the position of the element in the periodic system, no theoretical laws or suggestions have been expressed. It may be of interest to call attention to a few empirical rules which perhaps will give some guidance for later theoretical investigations.

1. A regularity which has been pointed out recently by Catalán[1] appears to be of great importance. By dividing the "transition series"[2] of elements between A and Cu into two groups of five as follows,

19 K	20 Ca	21 Sc	22 Ti	23 V
^2S	^1S	^2D	^3F	^4F
24 Cr	25 Mn	26 Fe	27 Co	28 Ni
^7S	^6S	^5D	^4F	^3F

there is brought into evidence for elements in each column the remarkable similarity of the positions of the terms in the energy diagrams and the identity of the azimuthal quantum numbers of the normal states.[3]

* Published by permission of the Director of the Bureau of Standards of the U. S. Department of Commerce.

[1] Catalán, Bay. Akad., p. 15, 7 Feb. 1925.

[2] Main Smith, Atomic Structure and Chemistry, London, 1924.

[3] We wish to add that this regularity confirms the lowest term of the iron spectrum as a D term. That the quintet D term discovered in the analysis of this spectrum and supported by the absorption experiments of Gieseler and Grotrian and of Angerer and Joos, really

Catalán also proposes the following arrangement for the first spark spectra,

19 Ca^+	20 Sc^+	21 Ti^+	22 V^+	23 Cr^+
^2S	^3D	^4F	^5F	^6D
24 Mn^+	25 Fe^+	26 Co^+	27 Ni^+	28 Cu^+
^7S	^6D	(^3F)	(^4F)	(^3D)

which is confirmed by the spectral classifications of the first two pairs, and predicts for Co^+, Ni^+ and Cu^+ structures similar to those observed for Ti^+, V^+ and Cr^+. The spectral structures for elements of the second long period are now well enough known to show that a similar division of the first ten, between Kr and Ag, into two groups of five is also justified.

2. The identification of the lowest levels in various spectra is, in general, supported by the classification of the *raies ultimes*, the most sensitive lines for spectrographically detecting the chemical elements. A simple rule which indicates the terms from which these lines arise was tentatively pointed out by Meggers, Kiess and Walters[4] about a year ago. In the meantime this rule has been found to hold for all the arc and spark spectra which have been sufficiently analysed. We submit the generalization that the *raies ultimes* are combinations of the lowest term with the first higher non-metastable term of the same series system, definite preference being shown[5] for $\Delta l = -1$ even if the energy difference is somewhat larger than for other strong combinations involving the lowest term, provided that this difference is not exceptionally great so that the preference is overwhelmed by a closer combination. In complex spectra, where there are usually many terms of approximately the same magnitude which according to the larger combination possibilities ($\Delta l = 1, \pm 3$ for isomorphous and $\Delta l = 0, \pm 2$ for heteromorphous terms) can combine with the lowest term, the *raies ultimes* nevertheless prefer the jump $\Delta l = -1$. The difference between the first terms which may combine with the lowest one in the case of the Al group is, however, so unusually large (7100 to 10,000 cm^{-1}) that the *raies ultimes* for these metals apparently must involve the jump $\Delta l = +1$ from the nearer term.

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represents the normal state, in spite of the results of the Stern-Gerlach experiment (Ann. der Phys., 379, 673; 1924) is further favored by many astrophysical facts and by the almost complete line classification which now makes it improbable that any important low terms are unknown.

[4] Meggers, Kiess and Walters, J.O.S.A. & R.S.I., *9*, p. 355; 1924.

[5] $\Delta l = -1$ because the electron loses one unit azimuthal quantum when going from its initial to the final state.

The above-mentioned rule is illustrated in Table 1 where the *raies ultimes* of the first ten elements of both long periods are given so far as known for arc and for first spark spectra. The wave lengths or corresponding excitation potentials of the spark lines when plotted for either period fall on a remarkably smooth curve, whereas the

TABLE 1. *Raies ultimes.*

		Arc		Spark	
Z	Element	Terms	λ	Terms	λ
19	K	$^2S_1 - {}^2P_2$	7664.94		
37	Rb		7800.29		
20	Ca	$^1S_0 - {}^1P_1$	4226.73	$^2S_1 - {}^2P_2$	3933.66
38	Sr		4607.34		4077.71
21	Sc	$^2D_2 - {}^2F_4$	4779.35	$^3D_3 - {}^3F_4$	3613.86
39	Y		4674.84		3710.30
22	Ti	$^3F_4 - {}^3G_5$	3635.47	$^4F_5 - {}^4G_6$	3349.41
40	Zr		3519.60		3391.96
23	V	$^4F_5 - {}^4G_6$	3185.40	$^5F_5 - {}^5G_6$	3093.10
41	Cb		?		3094.20
24	Cr	$^7S_3 - {}^7P_4$	4254.34	$^6D_5 - {}^6F_6$	2835.64
42	Mo		3798.26		2816.16
25	Mn	$^6S_3 - {}^6P_4$	4030.76	$^7S_3 - {}^7P_4$	2576.12
43	—		—		—
26	Fe	$^5D_4 - {}^5F_5$	3719.94	$^6D_5 - {}^6F_6$	2382.04
44	Ru		3498.95		?
27	Co	$^4F_5 - {}^4G_6$	3453.51	$(^5F_5 - {}^5G_6)$	2388.93
45	Rh		3434.90		?
28	Ni	$^3F -$	3414.77	$(^4F_5 - {}^4G_6)$	2416.16
46	Pd		3634.69		?

analogous curve for the arc is quite irregular. It seems therefore that under the influence of the higher effective nuclear charge the quantitative relations of the energy diagrams in the transition series become simpler.

TABLE 2. *Resonance lines*

		Arc		Spark	
Z	Element	Terms	λ	Terms	λ
19	K	$^2S_1-{}^2P_2$	7664.94		
37	Rb		7800.29		
20	Ca	$^1S_0-{}^3P_1$	6572.78	$^2S_1-{}^2P_2$	3933.66
38	Sr		6892.62		4077.71
21	Sc	$(^2D_3-{}^4F_4)$	>6413.	$^3D_3-{}^3F_4$	3613.86
39	Y		>6933		3710.30
22	Ti	$^3F_4-{}^3G_5$	6295.30	$^4F_5-{}^4G_6$	3349.41
40	Zr		?		3391.96
23	V	$^4F_5-{}^6D_4, {}^6G_6$?	5632.47; ?	$^5F_5-{}^5G_6$	3093.10
41	Cb		?		3094.20
24	Cr	$^7S_3-{}^7P_4$	4254.34	$^6D_5-{}^6F_6$	2835.64
42	Mo		3798.26		2816.16
25	Mn	$^6S_3-{}^8P_4$	5394.68	$^7S_3-{}^7P_4$	2576.12
43	—		—		—
26	Fe	$^5D_4-{}^7\overline{D}_5, {}^7F_5$	5166.29; 4375.93	$^6D_5-{}^6F_6$	2382.04
44	Ru		?		?
27	Co	$^4F_5-{}^5\overline{F}_6, {}^5G_6$		$(^3F_5-{}^5G_6)$	2388.93
45	Rh		?		?
28	Ni	$(^3F_4-{}^5\overline{F}_5, {}^5G_5)$		$(^4F_5-{}^4G_6)$	2416.16
46	Pd		?		?

3. Another regularity which may be generalized from recent spectral classifications deals with the electron jump which involves the lowest

term and gives rise to the line of greatest wave length, i.e., the first transition which is connected with emission and requires the least energy for excitation. In spectra where there is no metastable level lying close to the lowest term these lines are the *resonance lines*. Although in nearly all the spectra to be dealt with in this section the lowest level has many neighboring metastable levels which prevent true resonance from taking place, for simplicity we shall call the "first excitable lines" resonance lines.

Our statement is that the above defined resonance line in a spectrum is a combination (usually with $\Delta l = -1$) of the lowest term with a term of the highest multiplicity (largest value of r or j_a) in the spectrum. If the normal state of the atom is itself described by a term belonging to the system of highest multiplicity the resonance line coincides with the *raie ultime*; if the lowest term belongs to a system of lesser multiplicity the resonance line is an inter-system combination. The former is generally the case for spark spectra while the latter is nearly always true for arc spectra. In Table 2 we give the evidence for this rule, so far as known for the spectra discussed in section 2 or Table 1.

It appears that the multiplicity of the term combining with the lowest one for a resonance line is always the highest of the spectrum; it increases uniformly for the first seven spectra and then decreases for the last three. The resonance lines in the arc spectra are always inter-system combinations except for K and Cr (Rb and Mo) which are indeed supposed to have a significant similarity according to Catalán's relation described in section 1. For the iron (and ruthenium) triad a combination $\Delta l = 0$ actually gives the first excitable line although the first $\Delta l = -1$ combination with the lowest term is in the same region of the spectrum.

It is noted that this rule is valid also for oxygen and sulphur. The resonance line for O and for S is an inter-system combination, $^3P_2 - {}^5S_2$; the wave length of this line for O is 1355.73 A.

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Note added at the correction of proof. On the basis of recent theoretical investigations by Pauli, Heisenberg and Hund (ZS.f. Phys. *31*, *32*, *33*, resp.) the following explanations may be given for our rules 2 and 3: (2) It is permissible to compare intensities of lines in different multiplets with each other provided that the terms involved correspond to the same values of azimuthal quantum numbers for the individual electrons. Consequently in complex spectra with close lying term groups (e.g. D, $\overline{F}$, G) the combinations with the term of largest l and j values is the strongest one. (3) The explanation for rule 3 seems to lie in the fact stated by Hund that among term groups with the same k values those with highest multiplicity are the lowest ones.

Observations on Point-Discharge.—Wulf found that a convenient way of observing point-discharges consists in connecting the point and the enclosing chamber to the two coatings of a charged Leyden jar. The charge on the Leyden jar "runs down" by virtue of a series of spontaneous point discharges to a threshold value of potential, where the spontaneous discharges cease; from this stage onward, discharges can be evoked by alpha particles passing into the chamber, until the potential sinks past a second threshold value. The interval between these two thresholds, which serves as a measure of the value of the device as an alpha-particle counter, depends on many factors; it lies higher for dull "points" than for sharp ones, higher for positively charged points than for negatively charged ones, higher for benzine and CO_2 than for air or hydrogen as the gas filling the chamber, higher for gas at atmospheric than for gas at a lower pressure; and in general, the higher the interval lies the wider it is. However, it appears to be unaffected by moisture in the air, which seems odd, and by the direction in which the alpha particles pass through the chamber, which contradicts Kutzner's result (This Journal, *9*, p. 674). The discharges are individually visible, and show characteristic patterns of light and dark spaces depending on the polarity of the point, somewhat but not altogether like those observed by Zeleny (This Journal, *10*, p. 168). If the discharges are excited too frequently, say more than 20 times per second, they fuse into a continuous discharge if the point is positive, but not if it is negative. The author feels that the reason for the cessation of the discharge provoked by an alpha-particle is the same as the reason for the spontaneous cessation of the spontaneous discharge, whatever that may be. The article is unusually well written.—[T. Wulf, Valkenburg; Phys. ZS. *26*, pp. 382-391; 1925.] KARL K. DARROW

Extraction-Potentials for the outer Electrons of Elements from 50 to 74.—Measurements are made upon the *L* absorption frequencies, and then the *M*, *N*, and *O* absorption frequencies are calculated by combining those with emission-frequencies recorded by other observers. These frequencies, translated into volts, represent the energies required to remove electrons from the various shells of the atom. Plotting the square roots of these energies versus the atomic number, one gets curves which have "breaks" near the beginning and near the end of the group of rare earth elements; these breaks are quite inconspicuous for the inner electrons, progressively more noticeable for those lying farther out. This is taken as evidence for Bohr's idea that the different rare earth elements differ in the number of electrons in not the outermost but the next to the outermost layer. The photographs show a set of quite distinct absorption lines instead of the familiar sharp absorption edge, which, the author thinks, results from careful adjustment of the thickness of the absorbing stratum.—[Y. Nishina; Phil. Mag. *49*, pp. 521-536; 1925.]

KARL K. DARROW

DEPOLARIZING INFLUENCE OF ALTERNATING MAGNETIC FIELDS ON RESONANCE RADIATION

By G. Breit

It has been shown by Fermi and Rasetti[1] as well as by Ellett[2] and the writer that the depolarizing influence of an alternating magnetic field on the resonance radiation of mercury is at times considerably smaller than the influence of an equal field applied at a low frequency. The calculations below refer to a classical isotropic oscillator the behavior of which has been discussed for steady fields before.[3]

We suppose the electric intensity of the incident wave to be directed along the axis OX and the direction of the externally applied magnetic field to be along OZ. The applied magnetic field we suppose to start with to be a periodic function of the time of the form $H\cos pt$. The equations of motion for the isotropic oscillator may be written

$$\ddot{x}+\kappa\dot{x}+\Omega^2 x=\frac{eE_z}{m}\cos(\omega t-\psi)+\frac{eH}{mc}\left(\dot{y}\cos pt-\frac{py}{2}\sin pt\right) \tag{1}$$

$$\ddot{y}+\kappa\dot{y}+\Omega^2 y=-\frac{eH}{mc}\left(\dot{x}\cos pt-\frac{px}{2}\sin pt\right) \tag{2}$$

In these equations the terms $\frac{eH}{cm}\dot{y}\cos pt, -\frac{eH}{mc}\dot{x}\cos pt$ represent the force on the moving electron due to the magnetic field $H\cos pt$. The terms $-\frac{eHp}{2mc}y\sin pt$ and $+\frac{eHp}{2mc}x\sin pt$ take account of the induced electromotive force due to variations in the magnetic field $H\cos pt$ on the supposition that the variations in the magnetic field cause no electric intensity at the position of equilibrium of the electron. This supposition means that an electric field varying with the time but uniform in space is neglected. The frequency of the electric field being the low frequency $p/2\pi$, it is justifiable to neglect it for the isotropic oscillator because its only effect is to superpose a vibration of this frequency which has no observable optical effect in the region of the spectrum corresponding to ω.

[1] Nature, London, p. 764, May 16, 1925.
[2] American Physical Society meeting of April 24-25, 1925.
[3] J. O. S. A. & R. S. I. *10*, p. 439, 1925.

It will be convenient now to employ the letter u for

$$x+iy=u\ ,\ i=\sqrt{-1} \tag{3}$$

which enables us to write (1) and (2) in a single equation

$$\ddot{u}+(\kappa+2ip_o\cos pt)\dot{u}+(\Omega^2-ipp_o\sin pt)u=\frac{eE_x}{m}\cos(\omega t-\psi) \tag{4}$$

where $p_o=\dfrac{eH}{2mc}$. This is a linear differential equation of the second order. If we write

$$v=u\exp\left[\tfrac{1}{2}\int(\kappa+2ip_o\cos pt)dt\right] \tag{4.1}$$

we obtain

$$\frac{d^2v}{dt^2}+[\Omega^2-\tfrac{1}{4}(\kappa+2ip_o\cos pt)^2]v=\frac{eE_x}{m}\cos(\omega t-\psi)\exp\left[\frac{\kappa t}{2}+i\frac{p_o}{p}\sin pt\right] \tag{4.2}$$

If p_o^2 is neglected we deal with an equation the left-hand member of which is of the Mathieu type. If in addition $\kappa=o$, we obtain the simple form

$$\frac{d^2v}{dt^2}+\Omega^2v=\frac{eE_x}{m}\cos(\omega t-\psi)\exp\left(i\frac{p_o}{p}\sin pt\right) \tag{4.3}$$

where

$$v=u\exp\left[i\frac{p_o}{p}\sin pt\right] \tag{4.4}$$

Equation (4.3) is very much easier to solve than (4.2). However, physically it is not satisfactory inasmuch as we leave out κ in it and κ is very important in determining the whole behavior of the polarization of resonance radiation.

If now we deal (as is usually the case) with such fields that p_0 and κ may be of the same order of magnitude but that both are small compared with Ω, then we may write instead of (4.2)

$$\frac{d^2v}{dt^2}+\Omega^2v=\frac{eE_x}{m}\cos(\omega t-\psi)\exp\left[\frac{\kappa t}{2}+i\frac{p_o}{p}\sin pt\right] \tag{4.5}$$

with an accuracy which implies the neglect of the second order Zeeman effect in comparison with the first.

A solution of this equation (4.5) is

$$v=\int_{-\infty}^{t}\frac{eE_x}{m\Omega}\cos(\omega\xi-\psi)\exp\left[\frac{\kappa\xi}{2}+i\frac{p_o}{p}\sin p\xi\right]\sin\Omega(t-\xi)d\xi \tag{5}$$

Making the substitution

$$t-\xi=\eta \tag{5.1}$$

and using (4.1) we have

$$u=\frac{eE_x}{2m\Omega}\int_0^\infty \exp\left(-\frac{\eta\kappa}{2}-i\frac{2p_0}{p}\sin\frac{p\eta}{2}\cos p\left(t-\frac{\eta}{2}\right)\right)\left[\sin\{(\Omega-\omega)\eta+\omega t-\psi\}\right.$$
$$\left.+\sin\{(\Omega+\omega)\eta-\omega t+\psi\}\right]d\eta \tag{6}$$

By means of (3) we can now obtain x and y as

$$x=S_x\sin(\omega t-\psi)+C_x\cos(\omega t-\psi) \tag{7.1}$$

$$y=S_y\sin(\omega t-\psi)+C_y\cos(\omega t-\psi) \tag{7.2}$$

where

$$S_x=\frac{eE_x}{2m\Omega}\int_0^\infty \exp\left(-\frac{\eta\kappa}{2}\right)\cos\gamma\,[\cos(\Omega-\omega)\eta-\cos(\Omega+\omega)\eta]d\eta \tag{7.3}$$

$$C_x=\frac{eE_x}{2m\Omega}\int_0^\infty \exp\left(-\frac{\eta\kappa}{2}\right)\cos\gamma\,[\sin(\Omega-\omega)\eta+\sin(\Omega+\omega)\eta]d\eta \tag{7.4}$$

$$S_y=-\frac{eE_x}{2m\Omega}\int_0^\infty \exp\left(-\frac{\eta\kappa}{2}\right)\sin\gamma\,[\cos(\Omega-\omega)\eta-\cos(\Omega+\omega)\eta]d\eta \tag{7.5}$$

$$C_y=-\frac{eE_x}{2m}\int_0^\infty \exp\left(-\frac{\eta\kappa}{2}\right)\sin\gamma\,[\sin(\Omega-\omega)\eta+\sin(\Omega+\omega)\eta]d\eta \tag{7.6}$$

with the abbreviation

$$\gamma=\frac{2p_0}{p}\sin\left(\frac{p\eta}{2}\right)\cos p\left(t-\frac{\eta}{2}\right) \tag{7.7}$$

Here S_x, C_x, S_y, C_y are periodic functions of t with period $2\pi/p$. One can show without difficulty that the time averages of x^2 and y^2 are

$$\overline{x^2}=\tfrac{1}{2}(\overline{S_x^2}+\overline{C_x^2})\,,\quad \overline{y^2}=\tfrac{1}{2}(\overline{S_y^2}+\overline{C_y^2}) \tag{8}$$

Thus our problem is reduced to obtaining the right-hand values of (8). Having done so we must let ω vary and integrate over the spectrum of incident ω's. In doing this it is found that terms in $\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}(\Omega+\omega)\eta$ are of little importance because $(\Omega+\omega)$ is very high compared with p. The integration over the spectrum is then effected quite easily if we remember that if

$$f(t)=\int_0^\infty (A(u)\sin ut+B(u)\cos ut)du \tag{8.1}$$

then

$$\int_{-\infty}^{+\infty}(f(t))^2dt=\pi\int_0^\infty[(A(u)^2+(B(u))^2]du \tag{8.2}$$

We thus find that

$$\int_0^\infty S^2{}_x d\omega=\int_0^\infty C^2{}_x d\omega=\pi\left(\frac{eE_x}{2m\Omega}\right)^2\int_0^\infty(\exp(-\eta\kappa))\left[J^2{}_0\left(\frac{2p_0}{p}\sin\frac{p\eta}{2}\right)\right.$$
$$\left.+2J_2{}^2(\ \ )+2J_4{}^2(\ \ )d\eta\right.$$

$$\int_0^\infty S_y{}^2 d\omega=\int_0^\infty C_y{}^2 d\omega=\pi=\left(\frac{eE_x}{2m\Omega}\right)^2=\int_0^\infty(\exp(-\eta\kappa))\left[2J_1{}^2\left(\frac{2p_0}{p}\sin\frac{p\eta}{2}\right)\right.$$
$$\left.+\cdots\right]d\eta$$

where J_n is the Bessel function of the first kind and order n. Hence using the formula

$$J_0(2Z)=J_0{}^2(Z)+2J_1{}^2+2J_2{}^2+\cdots \tag{8.3}$$

we find that the average polarization

$$P=J_0{}^2\left(\frac{2p_0}{p}\right)+\frac{2J_1{}^2\left(\frac{2p_0}{p}\right)}{1+\frac{p^2}{\kappa^2}}+\frac{2J_2{}^2\left(\frac{2p_0}{p}\right)}{1+\frac{4p^2}{\kappa^2}}+\frac{2J_3{}^2\left(\frac{2p_0}{p}\right)}{1+\frac{9p^2}{\kappa^2}}+\cdots \tag{9}$$

It may be mentioned that the same result is arrived at by a much more laborious method if instead of using (4.5) we use (4.3) and consider the effect of interrupting the vibrations in the well-known manner introduced by Rayleigh and Lorentz. If this is done, terms in $\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}(\Omega+\omega)\eta$ are absent, however, in equations (7.3) . . . . (7.6). This calculation is too laborious to make its publication worth while. The above formula (9) gives the degree of polarization on the supposition that the classical isotropic oscillator is illuminated by a band of frequencies which is wide in comparison with the width of the absorption band of the oscillator. We can now compare this result with that derived on the hypothesis of Eldridge. On that hypothesis for mercury the excitation may be imagined to take place when a certain axis in the atom is parallel to OX. Under the influence of the Larmor precession the atom turns and when it is ready to emit its emission axis is inclined to the axis of X. Thus a depolarization is produced.

At the time t_0 a certain number of atoms are excited. At the time t the axes of these atoms have turned through an angle

$$\alpha = \int_{t_0}^{t} p_0 \cos pt \, dt = \frac{p_0}{p}[\sin pt - \sin pt_0] \tag{10}$$

The probability of emission by this group of atoms in the interval $t, t+dt$ is $e^{-\frac{t-t_0}{\tau}} dt$ where τ is the mean life of the atom. The contributions to the intensities along OX and OY are then

$$X(t_0) = \int_{t_0}^{\infty} e^{-\frac{t-t_0}{\tau}} \cos^2 \frac{p_0}{p}(\sin pt - \sin pt_0) dt \tag{11.1}$$

$$Y(t_0) = \int_{t_0}^{\infty} e^{-\frac{t-t_0}{\tau}} \sin^2 \frac{p_0}{p}(\sin pt - \sin pt_0) dt \tag{11.2}$$

These values must further be averaged over t_0.

We have from (11.1), (11.2) that

$$X(t_0) - Y(t_0) = \int_{t_0}^{\infty} e^{-\frac{t-t_0}{\tau}} \cos \frac{2p_0}{p}(\sin pt - \sin pt_0) dt$$

$$X(t_0) + Y(t_0) = \int_{t_0}^{\infty} e^{-\frac{t-t_0}{\tau}} dt = \tau$$

It is found in this way that

$$P = J_0^2\left(\frac{2p_0}{p}\right) + \frac{2J_1^2\left(\frac{2p_0}{p}\right)}{1+p^2\tau^2} + \frac{2J_2^2\left(\frac{2p_0}{p}\right)}{1+4p^2\tau^2} + \cdots \tag{12}$$

thus a result somewhat similar to (9) but not identical with it inasmuch as

$$\tau = \frac{2}{\kappa}$$

This difference, however, is to be expected because a precisely similar difference exists between the formula (1) of Eldridge[4] and the writer's expression (4.1).[5]

[4] Eldridge, J. A., Physic. Rev. *24*, p. 234; 1924.
[5] Breit, G., J. O. S. A. & R. S. I. *10*, p. 439; 1925.

Since (9) and (12) become identical if we substitute in (9) τ for $\frac{1}{\kappa}$ we see that for the isotropic oscillator we may use a quantity

$$P(t_o)=\frac{1}{\tau}\int_{t_o}^{\infty} e^{-\frac{t-t_o}{\tau'}}\cos\frac{2p_o}{p}(\sin pt-\sin pt_o)dt \tag{14}$$

where

$$\tau'=\frac{1}{\kappa}=\frac{\tau}{2} \tag{14.5}$$

and

$$P=\overline{P(t_o)} \tag{15}$$

This form of the answer is useful if p is small compared to $\frac{1}{\tau}$ and to p_o. We find by means of it that the limiting value of P is

$$(P)_{p=o}=\frac{1}{\sqrt{1+\omega^2\tau^2}} \tag{16}$$

The relation between the model of Eldridge and the classical isotropic oscillator which we have found is, of course, not limited to simple harmonic vibrations in H but is universal. For if

$$H(t)=H_o f(t) \tag{17}$$

we obtain a set of equations such as (7.3), (7.4), (7.5), (7.6), the only difference being that instead of (7.7) we now have the more general expression

$$\gamma=p_o\int_{t-\eta}^{t} f(x)dx \tag{18}$$

An application of (8.2) introduces expressions of the type $\exp(-\eta\kappa)\left\{\begin{matrix}\cos^2\\ \sin^2\end{matrix}\right\}\gamma$ which would be also obtained in the same way from the model of Eldridge if in Formulas (11) and (12) we write $\tau=\frac{1}{\kappa}$, $\eta=t-t_o$.

It is thus clear that in general the problem of finding the degree of polarization can be always treated by the model of Eldridge with a proper understanding about τ.

Thus if a steady field is applied to the alternating one we get for the polarization

$$P=\frac{2}{\tau}\int_0^\infty e^{-\frac{2u}{\tau}}\cos(2vu)\left[J_0^2\left(\frac{2p_0}{p}\right)+2J_1^2\left(\frac{2p_0}{p}\right)\cos pu+\cdots\right]du$$

$$=\frac{J_0^2\left(\frac{2p_0}{p}\right)}{1+\left(2v\frac{\tau}{2}\right)^2}+J_1^2\left(\frac{2p_0}{p}\right)\left(\frac{1}{1+\left(\frac{\tau}{2}\right)^2(p+2v)^2}+\frac{1}{1+\left(\frac{\tau}{2}\right)^2(p-2v)^2}\right)$$

$$+J_2^2\left(\frac{2p_0}{p}\right)\left(\frac{1}{1+\left(\frac{\tau}{2}\right)^2(2p+2v)^2}+\frac{1}{1+\left(\frac{\tau}{2}\right)^2(2p-2v^2)}+\cdots\right) \quad (19)$$

where v is the value of the Larmor precession corresponding to the steady field. If here $p_0=0$, we obtain $(1+v^2\tau^2)^{-1}$ in agreement with the result for a steady field alone.

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Standardization of Colors for traffic Signals.—The work involved in bringing about national uniformity in the use of colors for traffic signals is nearing its completion; full agreement has been reached upon the various technical details, and the code is about to be published. The work has been carried out by a sectional committee on which all interested groups are represented, including more than thirty national organizations, under the leadership of the American Association of State Highway Officials, the Bureau of Standards, and the National Safety Council.

The code covers the use of luminous and non-luminous signs and signals in connection with highway traffic, including moving and flashing signals; the use of lights, semaphores, and other signaling devices on vehicles. The three colors agreed upon for primary traffic control signals are: red—for stop; yellow—for caution; and green—to proceed. "The use of red is proper as an indication to stop and to then proceed if conditions are favorable, as, for example, when "stop" and "proceed" regulations are in effect. Yellow is appropriate when caution is to be exercised without stopping, as for partial street obstruction, so as to reserve red for a stop signal. "Green shall be used as an indication to proceed."

The code gives concise qualitative definitions of these three colors as well as of colored glasses and of the colors recommended for non-luminous signs. These recommendations are based on the findings of three subcommittees which collected exhaustive data on the present diversity of highway signals, made a thorough study of the questions

relating to visibility of colors and also to sign boards and other non-luminous highway signals.

Careful experiments showed that the red signal lights were most easily distinguished from other colors at a distance and require the lowest light intensity for unmistakable recognition. On the average a red light of 75 candle-power could be identified at 600 feet, while a green light had to be of 250 candle-power, a yellow 750, and a blue light 1,000.

It is of paramount importance that the use or significance of all these signals become so familiar that they will produce an unconscious but correct and effective reaction, as it is doubt in the correctness of signal interpretation that leads to uncertainty in motions and to accidents.

AMERICAN ENGINEERING STANDARDS COMMITTEE,
29 WEST 39TH ST., NEW YORK, N. Y.

Illuminating engineering Nomenclature and photometric Standards.—A revision of the Illuminating Engineering Nomenclature and Photometric Standards has been approved by the American Engineering Standards Committee as "American Standard." The work includes the definition of terms used in illuminating engineering and photometry, together with the formulation of general principles to govern the measurement of light and illumination and the application of such measurements in practice. The revised standard includes a few additional terms and has a different statement for a number of the definitions.

Extensive discussion was given to units of brightness. In view of the standing which has been attained internationally by the "lambert," which is the unit for brightness, and in view of the usefulness of this term, its definition, together with a number of other definitions of units used in light measurements, have been retained as previously defined. Conversion factors for various working units are given.

This work, which is under the sponsorship of the Illuminating Engineering Society, was carried out by a sectional committee containing representatives of the Society, the American Gas Association, American Institute of Electrical Engineers, Bureau of Standards, U. S. National Committee of the International Commission on Illumination, U. S. National Committee of the International Electrotechnical Commission, National Council of Lighting Fixture Manufacturers, National Electric Light Association, Optical Society of America, together with independent experts. The Chairman of the Committee is Mr. E. C. Crittenden of the Bureau of Standards.

AMERICAN ENGINEERING STANDARDS COMMITTEE,
29 WEST 39TH ST., NEW YORK. N. Y.

SPECTRAL CENTROID RELATIONS FOR ARTIFICIAL DAYLIGHT FILTERS*

By K. S. Gibson

At the Boston meeting of the Optical Society of America, October 23-25, 1924, the author gave a paper under the above title, showing that[1] "For a given light source a linear relation exists between the thickness of the glass and the spectral centroid[2] of light transmitted by it," and "that for a given filter a linear relation exists between the spectral centroids of light at various color temperatures and of the same light transmitted by the filter," i.e., between the spectral centroids of the incident and transmitted light. The extent to which these relations had been tested and the precision with which they hold were illustrated.

It was expected at that time to publish these data along with much other material under the general subject of artificial daylight filters. A study of such filters has been made intermittently at the Bureau for several years and it is hoped that the contemplated paper may eventually be issued. It is believed, however, that the spectral centroid relations found are of sufficient practical value to justify their illustration at this time, prior to the publication of the complete paper.

VARIATION OF SPECTRAL CENTROID OF TRANSMITTED LIGHT WITH THICKNESS OF GLASS.

The study of these relations was made in connection with a test[3] of some Macbeth daylight wedges. A calibration of two pairs of these wedges in terms of color temperature[4] for a light source of 2848° K gave the curves shown in Fig. 1, where color temperatures on a logarith-

* Published by permission of the Director, Bureau of Standards.

[1] Proceedings, J.O.S.A. & R.S.I., *10*, p. 292; Mar. 1925.

[2] By spectral centroid, λ_c, is meant the wave length center of gravity of the luminosity curve: viz.,

$$\lambda_c = \frac{\Sigma V_\lambda E_\lambda T_\lambda \lambda}{\Sigma V_\lambda E_\lambda T_\lambda},$$

where $V_\lambda \equiv$ relative visibility,
$E_\lambda \equiv$ relative energy distribution of light source,
$T_\lambda \equiv$ spectral transmission of glass,
$\lambda \equiv$ wave length.

[3] No. 40871, July, 1924.

[4] Measurements were made on the rotatory dispersion colorimetric photometer designed by Priest: J.O.S.A. & R.S.I., 7, pp. 1175-1209, Dec. 1923.

mic scale are plotted against differences in wedge readings. The two wedges of any pair, exactly alike, are mounted with their thin edges in opposite directions, the light thus passing through a uniform thickness of glass for any position of wedges; and the abscissas of Fig. 1, reading from left to right, are proportional to the thickness of the daylight glass.

The location of the wedge readings corresponding to zero thickness of glass and to the color temperature of the light source alone was made possible by spectrophotometric observations from a previous test[5] on

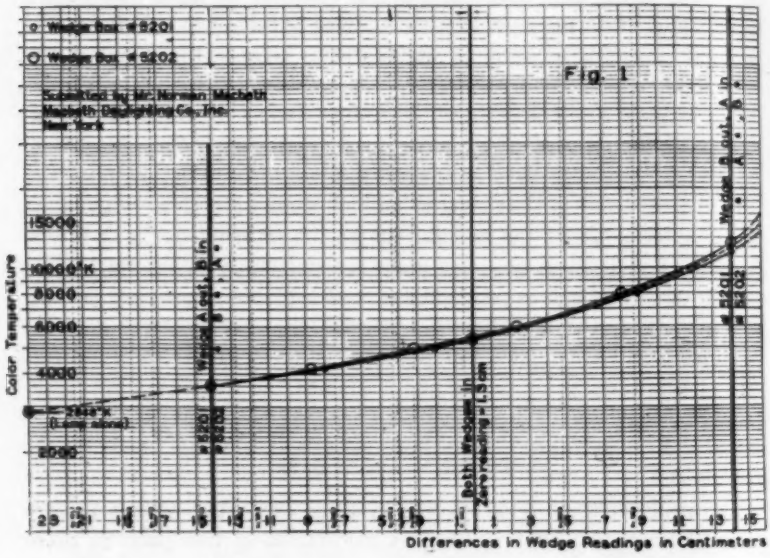

FIG. 1. *Showing relation between color temperature and thickness of daylight glass.*

the wedges of box No. 5201. By plotting transmissions at certain wave lengths on a logarithmic scale against differences in wedge readings (proportional to thickness of glass) in the usual way, Fig. 2, the wedge reading corresponding to zero thickness may be located at the intersection of the curves, the transmission at this intersection being equal to unity minus the fraction of light lost by reflection at the four surfaces.

When the data of Fig. 1 are plotted as in Fig. 3, with spectral centroid, λ_c, as ordinate instead of the logarithm of color temperature, the *linear relation* noted is obtained. The deviations of the plotted values

[5] No. 33652, 1st Supplement, July, 1922. Data by Dr. M. K. Frehafer.

from the straight lines as drawn vary from 0.0 to 0.3 mμ, not larger than possible observational error, and there seems to be no evidence of any consistent curvature.

The data so far presented are for Gage-Corning daylight glass. The relation was also tested in the case of Ives-Brady daylight glass by measuring the spectral centroids through two pieces separately and in combination. If the linear relation holds, the respective λ_c differences (light alone and light through filter) should add up to the difference when combined. In one case, the respective experimental differences,

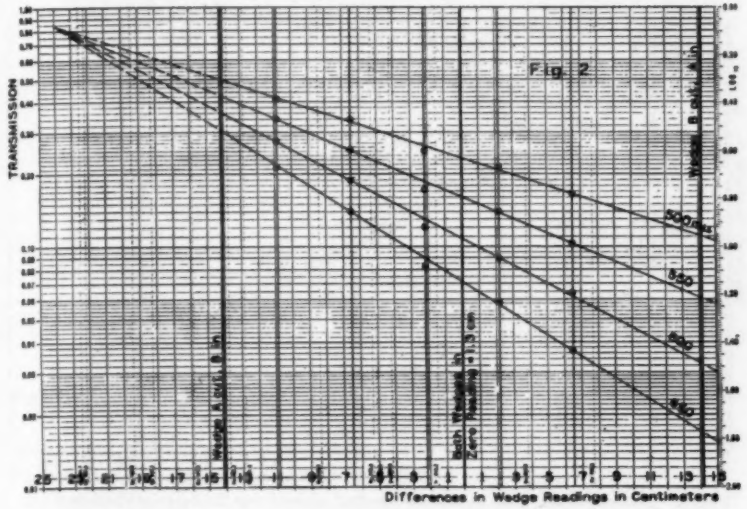

FIG. 2. *Illustrating method of locating wedge reading corresponding to zero thickness of glass.*

8.21 and 12.28 mμ, totalled 20.49 mμ, compared with 21.23, the experimental difference of the combination; in another case, the separate values totalled 20.21 instead of the experimental value of 20.95. The average deviation, about 3.5% (0.74/21.0), is perhaps outside of observational error; but the relation holds to a close approximation considering the range of color temperatures involved in this case, 2848° to about 16,000° K.

We may formulate the above evidence in a simple equation, as follows:

Let $\lambda_{c(s)}$ = spectral centroid of light source alone,

$\lambda_{c(1)}$ = spectral centroid of light source when modified by daylight glass of thickness b_1, and

$\lambda_{c(2)}$ = spectral centroid for some other thickness b_2.

Then:

$$\frac{\lambda_{c(s)} - \lambda_{c(1)}}{\lambda_{c(s)} - \lambda_{c(2)}} = \frac{b_1}{b_2}$$

This is probably a general relation holding for the usual sources and filters, but has actually been tested only for a source of light having a value of λ_c equivalent to 2848° K and for two kinds of daylight glass. The range of color temperatures covered, however, was from that of the source alone (2848° K) to that of blue sky (12,000°-16,000° K).

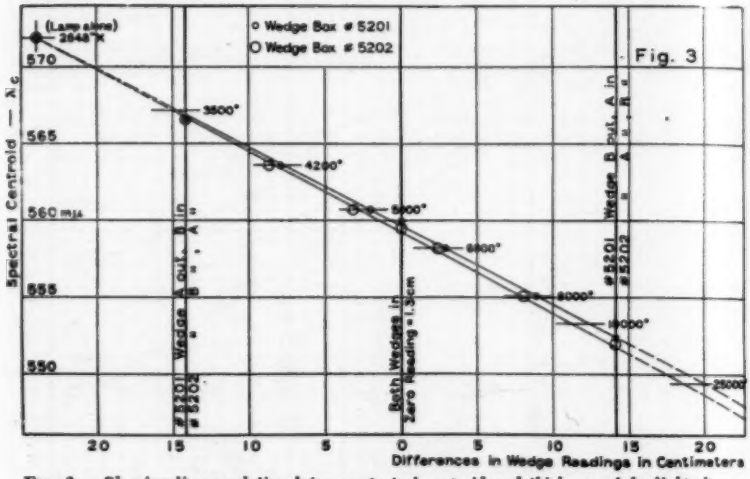

FIG. 3. *Showing linear relation between spectral centroid and thickness of daylight glass.*

This relation, even if approximate only, is of immediate value in the computation of the proper thickness to give a certain desired color temperature when used with a given source. Having such a source at a known color temperature, a single determination through a given daylight roundel furnishes information as to the proper thickness to grind to obtain any other color temperature down to that of the source alone. Its application in the design of daylight wedge colorimeters is obvious.

VARIATION OF SPECTRAL CENTROID OF TRANSMITTED LIGHT WITH THAT OF INCIDENT LIGHT.

A considerable range of daylight color temperatures may be effected by a variation of the color temperature of the source itself without

varying the thickness of the daylight filter. A very *exact linear relation* is here found, as illustrated in Fig. 4 for various types of filters at various color temperatures of source. These are computed values, not subject to observational uncertainty as in the previous case. None of the values deviate from the straight lines as drawn by more than 0.02 mμ.

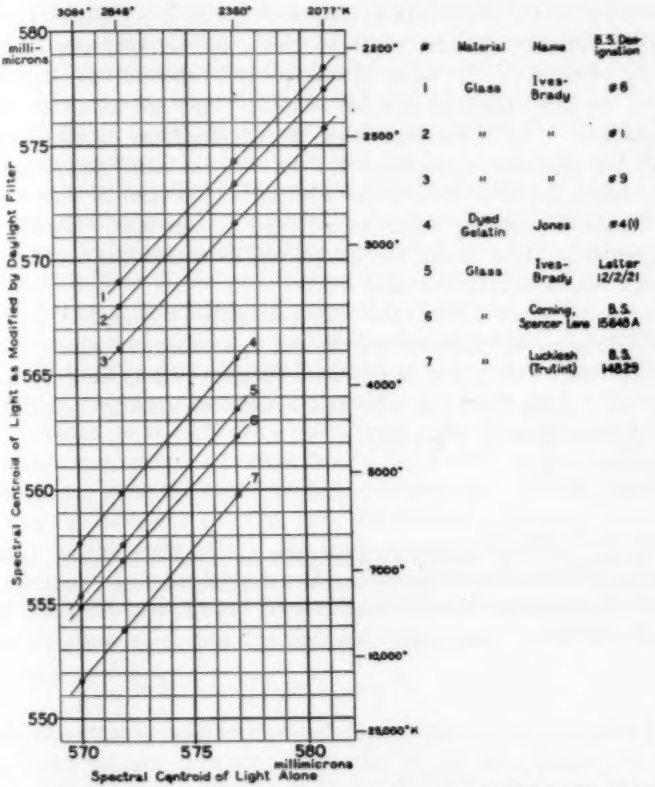

FIG. 4. *Showing linear relation between incident and transmitted light for daylight filters.*

Two determinations, therefore, are sufficient for accurate interpolation and for any reasonable extrapolation. In other words, the relation would seem to hold for all feasible incandescent sources when used with daylight filters giving color temperatures intermediate between that of the source alone and that of blue sky.

It may be of interest to note that the relation also holds approximately in the case of saturated red, yellow, and green signal glasses.

For color temperatures of a source equal to 2000°, 2360°, and 2848° K, respectively, the deviations of the values from the straight lines (the data being plotted as in Fig. 4) were in six out of seven cases less than 0.1 mμ.

RELATION BETWEEN SPECTRAL CENTROID AND COLOR TEMPERATURE.

In order for the above relations to be of the most practical value, the relation between spectral centroid and color temperature must be readily available. The value of the spectral centroid will depend, of course, on the values of relative visibility that are adopted for the computation. In connection with the development of the rotatory dispersion colorimetric photometer, Priest has published[6] tables showing this relation for color temperatures between 1838° and 24,000° K with the values of visibility which he adopted as standard. Curves constructed from these tables will enable one to obtain the continuously varying function between these limits.

While, as noted above, the spectral centroid computed for any particular color temperature will depend upon the particular values of visibility used,[7] the effect of different visibility values becomes negligible when going from one color temperature to another by means of the relations given in this paper.

WASHINGTON, D. C.
JUNE 1, 1925.

[6] Loc. cit., Tables 1 to 4.

[7] The magnitude of the variation of λ_c with visibility in the case of the visibility curve and the luminosity curve at 2077°K may be noted by reference to Table 3, p. 175, of Bureau of Standards Scientific Paper No. 475, "The Visibility of Radiant Energy," K. S. Gibson and E. P. T. Tyndall.

ASPHERICAL LENS SYSTEMS*

By Ludwik Silberstein

In the present paper the properties of the well-known Cartesian surfaces are first briefly recalled, and then a general method is given for building up an aberrationless system of any number of such refracting surfaces, each of these introducing a new free parameter. In the next place it is shown how such parameters can be utilized for satisfying other requirements in addition to that of the freedom from aberration. As an example a method is worked out for fulfilling, by means of two Cartesian surfaces, the sine-condition up to fourth-order terms and thus constructing what is technically called an aplanatic lens. Lastly, the achromatisation of such a lens by the insertion of a third surface is dealt with in a general way and is illustrated by a numerical example.

A certain part of the results described in this paper is embodied in the U.S. Patent No. 1,507,212, granted September 2, 1924.

1. A Cartesian surface, sometimes called aplanatic, is a refracting (or reflecting) surface free from spherical aberration, rigorously but, of course, only for a given object-point. Let O be such a point, and O' its image, so that all rays which issue from O and are refracted at the Cartesian surface converge in O'. Let $l=OP$ be the length of the incident ray from the object up to the surface σ, $l'=PO'$ that of the refracted ray from σ to O' (Fig. 1), μ the refractive index of the medium placed on the same side of the surface with the object, and μ' that of the medium on the opposite side. Then the freedom from aberration, which is given directly by Fermat's principle of stationary, and in our case constant light-time (or reduced "light-path") will be expressed by the equation

$$\mu l+\mu' l'=\text{const}. \tag{1}$$

This, the simplest and most direct form of the equation, holds for any Cartesian surface, provided only the object and image are on its opposite sides, as in Fig. 1. If O and O' are on the same side of the surface, the equation is

$$\mu l-\mu' l'=\text{const}. \tag{1a}$$

If these equations are expanded, say, in terms of rectangular coordinates, as will be done in the sequel, they turn out to be, in general,

* Communication No. 243 from the Research Laboratory of the Eastman Kodak Company. Paper read in abridgment at the annual meeting of the Optical Society of America at Cleveland, October 26, 1923.

of the fourth degree, that is to say, the meridian curve of any Cartesian surface, which is manifestly a surface of revolution around OO' as axis, is an algebraic curve of the fourth degree. Only if O is removed to infinity (when we have an incident beam of parallel rays), does it become a conic or curve of second degree.

Before proceeding to any expansions of (1), or (1a), it may be well to verify directly that either of these equations satisfies the elementary law of refraction, $\sin i : \sin i' = \mu' : \mu$ at every point P. This can be

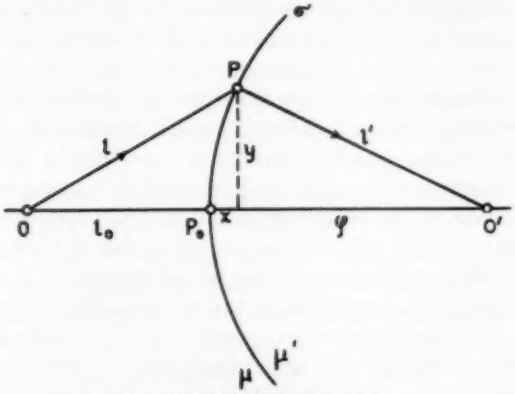

FIG. 1. *Single Cartesian surface.*

achieved vectorially with remarkable brevity. In fact, the normal to any surface $f = \text{const.}$ has the direction of the vector ∇f or grad f. Thus, in our case, if $\mathbf{n}$ be the unit normal of the surface (1) and a a scalar number,

$$a\mathbf{n} = \mu\nabla l + \mu'\nabla l'.$$

But $\nabla l = \mathbf{r}$ is a unit vector along OP and similarly $\nabla l' = \mathbf{r}'$ one along $O'P$. Thus,

$$a\mathbf{n} = \mu\mathbf{r} + \mu'\mathbf{r}'.$$

Multiply this vectorially by $\mathbf{n}$. Then, since $V\mathbf{nn} = 0$,

$$\mu V\mathbf{rn} = -\mu' V\mathbf{r}'\mathbf{n} \qquad (2)$$

which embodies Snell's formula $\mu \sin i = \mu' \sin i'$ and at the same time the coplanarity of $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{n}$ (through $\mathbf{r}V\mathbf{r}'\mathbf{n} = 0$), which is the complete law of refraction. Similarly for (1a).

Returning to Fig. 1, let $OP_0 = l_0$ be the axial value of l (or the shortest distance of O from σ), and $\varphi = P_0O'$, which is usually called "the section

distance" of the refracting surface. This fixes the value of the "constant" in (1) which will now become

$$\mu l+\mu' l' = \mu l_0+\mu'\varphi\,. \tag{3}$$

Similarly, *mutatis mutandis*, for (1*a*).

In particular, for O at infinity or for parallel incident rays, the length l can be reckoned from any wave-front (plane) up to the refracting surface in a direction parallel to the axis, say, from the plane tangential to the surface at P_0 (Fig. 2). Then $l_0=0$ and

$$\mu l+\mu' l' = \mu' f\,, \tag{4}$$

where f, the section distance corresponding to an object at infinity, will now be the focal length of the refracting surface.

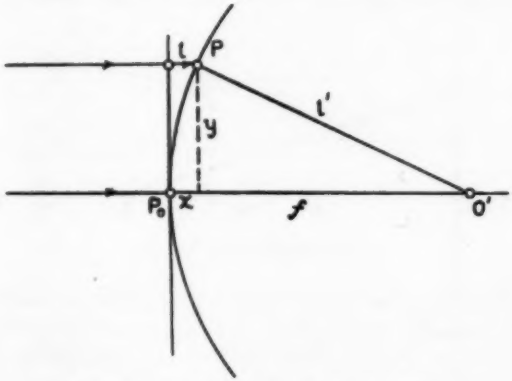

FIG. 2. *Sub-case for parallel incident rays.*

The equations of the meridian curves of these surfaces in terms of rectangular co-ordinates can be written down at once. If the co-ordinates of any point P are x, y, with OO' as x-axis and the axial point P_0 of the surface as origin, we have for the meridian curve of the surface (3)

$$\mu\sqrt{(l_0-x)^2+y^2}+\mu'\sqrt{(\varphi-x)^2+y^2}=\mu l_0+\mu'\varphi\,, \tag{3a}$$

which manifestly is of the fourth degree, and for that of the surface (4)

$$\mu x+\mu'\sqrt{(f-x)^2+y^2}=\mu' f\,, \tag{4a}$$

x being in both cases the distance of a point from the tangential plane

at P_0 and y that from the optical axis. The latter curve is easily seen to be a conic,

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1\,, \qquad \ldots\ldots\ldots\ldots \quad (4b)$$

with the semi-axes $a=\mu' f/(\mu'+\mu)$, $b=f\sqrt{\frac{\mu'-\mu}{\mu'+\mu}}$ and the eccentricity

$$\epsilon=\frac{\mu}{\mu'}\,.$$

Since in the present case (viz., of a collective surface, as in Fig. 2,) we have tacitly assumed $\mu'>\mu$, this conic will be an ellipse whose furthermost geometrical focus coincides with the optical focus O' and major

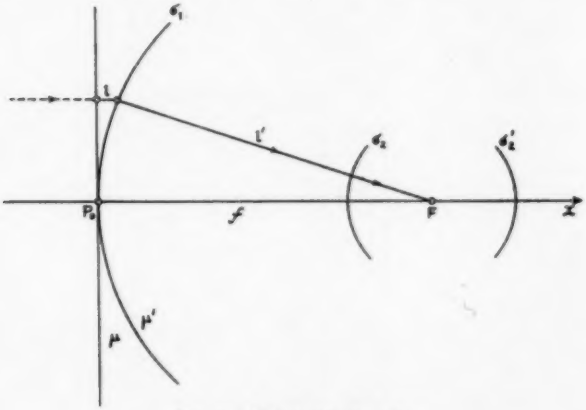

FIG. 3. *Spherical back surface.*

axis with the optical axis of the Cartesian surface. The alternative case of a dispersive surface,[1] either concave with $\mu'>\mu$ or convex but with $\mu'<\mu$, can be treated in much the same way.

2. Leaving these details on one side, we will now proceed to describe a simple but perfectly general method of constructing an aberrationless lens system of any number of refracting surfaces.

Let the first or the front-surface σ_1 of a lens be aberrationless for, say, an object at infinity, i.e., as in (4),

$$\mu l+\mu' l'=\text{const.}\ =\mu' f\,,$$

[1] Equation (4) is then to be replaced by $\mu l-\mu' l'=-\mu'\varphi$, where φ is still the distance of the focus from P_0.

where $l = x$ and $f = P_0F$ (Fig. 3), the previous focus O' being now denoted by F. To fix the ideas let there be air in front of and glass behind this surface. The glass has to be bounded by another, the back surface of the lens. Now, as far as my knowledge goes, the only known method of thus closing the lens consisted in introducing, theoretically, as back surface σ_2 a spherical surface centred at F, the focal point of the front surface σ_1. Since the rays refracted at σ_1 are all normal to the sphere σ_2 (no matter what its radius and whether its foremost or its back part σ_2' is utilized), they will pass through σ_2 unrefracted, so that the system will continue to be free from aberration, the point F being still the focus of the whole lens.

Such a method, however, of terminating or closing up a lens seems too narrow, in as much as it compels us to use for the second, and similarly, in more complex systems, for the third and all the following

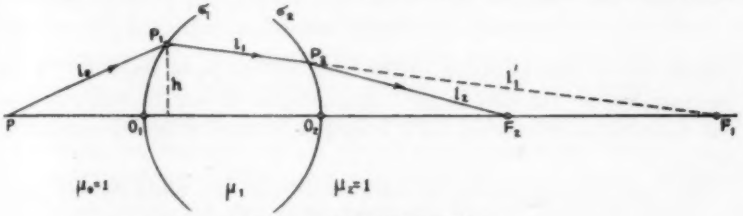

FIG. 4. *Two Cartesian surfaces.*

surfaces, spherical surfaces all centred at the same point F, and therefore not leaving a free choice even of their radii, independent, that is, of the glass thicknesses.

Instead of this a much more general method will here be proposed which will enable us to introduce with every further surface a new focus (instead of retaining the first one, F) and therewith also a new parameter. These free parameters will then be available for satisfying some further requirements in addition to the absence of aberration.

The conceptual side of the method which will now be described is exceedingly simple. It is, in fact, equally simple for three or more as for two surfaces. Yet, to fix the ideas, let us take first the case of *two* surfaces. At the same time, however, not to limit the treatment unnecessarily, let the object point (instead of being at infinity) be any point P on the optical axis or axis of symmetry. Let σ_1 and σ_2 (Fig. 4) be two surfaces of revolution limiting the lens whose refractive index is μ_1. For the sake of generality let us write μ_0, μ_2 for the refractive indices of the anterior and the posterior medium (though this be air in

practice). Let l_0, l_1, l_2 be the lengths of the light paths in the zeroth, the first, and the second medium, leading from the object P to the final image or focus F_2 on the axis. Then the condition for the absence of spherical aberration, for P as object point, is

$$\mu_0 l_0+\mu_1 l_1+\mu_2 l_2=\text{const.}\,, \tag{5}$$

where "constant" means, as before, a value common to all rays incident from P, independent of the height $y=h$ of incidence.

Such being the requirement for the whole lens, let us fix the shape of its front surface σ_1 by the equation

$$\mu_0 l_0+\mu_1(l_1+l_1')=\text{const.}\,, \tag{σ_1}$$

where l_1' is the (fictitious) continuation of the actual ray l_1 up to its intersection F_1 with the axis, as shown in Fig. 4. Now, subtract the last equation from (5), obtaining

$$-\mu_1 l_1'+\mu_2 l_2=\text{const.}\,, \tag{σ_2}$$

and let this be the equation of the back surface σ_2 of the lens. If the two surfaces be thus determined, the condition (5) for the whole system will manifestly be satisfied, for it is simply the sum of the two equations (σ_1), (σ_2).

The constants appearing in these equations can most conveniently be written in terms of the central thicknesses traversed by the axial ray $y=0$. Thus, if d_0 be the distance of the object P from the front surface, D the thickness of the lens, O_1 and O_2 the axial points of σ_1, σ_2, and $\varphi_1=O_1F_1$, $\varphi_2=O_2F_2$, we have as the complete equations of the two surfaces of the lens, assuming, with a view to practical cases, $\mu_0=\mu_2=1$ and putting $\mu_1=\mu$,

$$l_0+\mu(l_1+l_1')=d_0+\mu\varphi_1 \tag{σ_1}$$

$$\mu l_1'-l_2=\mu(\varphi_1-D)-\varphi_2\,. \tag{σ_2}$$

Combining these equations we have, of course, $l_0+\mu l_1+l_2=d_0+\mu D+\varphi_2$, which satisfies the final requirement.

The treatment of aberrationless lenses with three or more refracting surfaces will proceed on exactly the same lines. Thus, for instance, in the case of three surfaces σ_1, σ_2, σ_3 the final requirement is, with an obvious meaning of the symbols,

$$\mu_0 l_0+\mu_1 l_1+\mu_2 l_2+\mu_3 l_3=\text{const.} \tag{a}$$

To begin with, take for σ_1, as before,

$$\mu_0 l_0+\mu_1(l_1+l_1')=\text{const.}\,, \tag{a_1}$$

where l_1' is the straight continuation of l_1 up to the axis. Subtract (a_1) from (a), which gives

$$-\mu_1 l_1' + \mu_2 l_2 + \mu_3 l_3 = \text{const.}$$

Next, take for σ_2

$$-\mu_1 l_1' + \mu_2 (l_2 + l_2') = \text{const.} \qquad (a_2)$$

and finally, for σ_3, the difference of the last remainder and of (a_2), that is to say,

$$-\mu_2 l_2' + \mu_3 l_3 = \text{const.} \qquad (a_3)$$

The system composed of these three refracting surfaces will again satisfy the final condition (a), which is the sum of their equations, and will thus give a rigorously aberrationless image of the object P. The extension of this method to any number of surfaces is obvious. It will be enough to notice that any l_i' is defined with respect to l_i as was l_1' with respect to l_1.

That each of the surface equations thus obtained, as e.g. (a_1) or (a_2) in which appears either the sum or the difference of light-times, is in harmony with the refraction law, was already shown by a short vector method in the preceding section. It will be enough to recall that an equation of the type $\mu_1 l_1 - \mu_2 l_2 = \text{const.}$ gives the refraction law when the two fixed points, object and image, from which l_1 and l_2 are measured, are on the same side, while $\mu_1 l_1 + \mu_2 l_2 = \text{const.}$ gives the same law when the two fixed points are on the opposite sides of the surface defined by this equation.

This method can now be applied for building up an aberrationless system of any number of surfaces. It will be noticed that the central thickness of the medium between any two consecutive surfaces, as well as the positions of the points F_1, F_2, etc., the image points associated with each surface separately, can be freely chosen. These free parameters of the aberrationless system can be utilized for making the system satisfy some further optical requirements. An example of this kind will now be given.

3. *Two Surfaces. Method of satisfying the sine-condition. Aplanatic Lens.*—Let us consider a simple lens with only two refracting surfaces σ_1, σ_2, the former being used as a front surface. In the first place let it be required to make the lens free from aberration for an object at infinity or a beam of incident rays parallel to the axis. Then, if μ be the refractive index of the lens, the equations of the surfaces will be, as explained before,

$$l_0 + \mu(l_1 + l_1') = \text{const.} = \mu\overline{O_1F_1}$$

$$\mu l_1' - l_2 = \text{const.} = \mu\overline{O_2F_1} - \overline{O_2F_2},$$

or, in terms of x, y, with O_1 and O_2 (Fig. 5) as origin of x for the first and for the second equations respectively, and assuming a convex front surface,

$$x+\mu\sqrt{(\varphi_1-x)^2+y^2}=\mu\varphi_1 \tag{6}$$

and

$$\mu\sqrt{(f_1-x)^2+y^2}-\sqrt{(f_2-x)^2+y^2}=\mu f_1-f_2\,, \tag{7}$$

where

$$\varphi_1=O_1F_1\,,\quad \varphi_2=O_2F_2=f_2\,,$$
$$f_1=O_2F_1=\varphi_1-D\,,$$

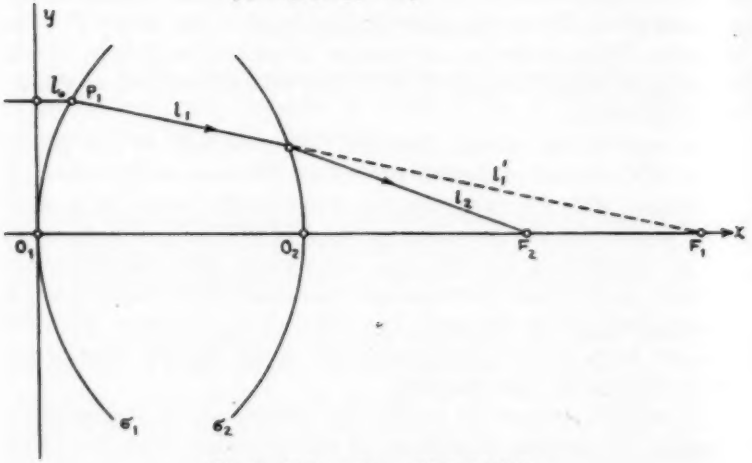

FIG. 5. *Sub-case for an object at infinity.*

D being the central thickness of the lens and f_1, f_2 the distances of the consecutive foci F_1, F_2 from the back surface. In common nomenclature F_2 would be called the (back) focus of the whole lens.

As we already saw, the meridian curve (6) of the front surface is an ellipse of eccentricity $1/\mu$ and with furthermost geometrical focus at F_1, while the meridian curve (7) of the back surface is of the fourth degree, as is in general the case for Cartesian surfaces.

Notice that even if φ_1 and μ be chosen and thus the first surface fixed, the parameter φ_2 of the second surface and the lens thickness D remain entirely free. In other words, the surfaces (6), (7) give an aberrationless lens *for any values* of φ_1, φ_2 as also of D and μ.

Such being the case, let us determine these available parameters so as to satisfy, approximately at least, *the sine-condition*, always for an object at infinity. A lens having this property and being at the same

time free from aberration would, in Abbe's nomenclature, be called a properly *aplanatic lens*.

The sine-condition for an object at infinity is

$$\frac{h}{\sin u'} = f = \text{const.}, \tag{8}$$

where f is the focal length of the lens, $y=h$ the height of an incident ray and u' the angle of inclination of the corresponding emergent ray to the axis. It remains to determine u' or $\sin u'$ as a function of the incidence height h, with (6) and (7) as the equations of the surfaces of the lens.

Let P_1 be the incidence point, x_1 its abscissa, with O_1 as origin, and $y_1=h$ its ordinate. Then the equation of the ray P_1F_1, with O_2 as origin of x, will be

$$\frac{x}{\varphi_1-D} + \frac{(\varphi_1-x_1)y}{(\varphi_1-D)h} = 1 . \tag{9}$$

This, with (7), will determine the co-ordinates x_2, y_2 of the point of the back surface hit by the ray. For x_1 in terms of h we have, by (6),

$$(\mu^2-1)x_1^2 - 2\mu\varphi_1(\mu-1)x_1 + \mu^2h^2 = 0 ,$$

whence the required root

$$x_1 = \frac{\mu\varphi_1}{\mu+1}\left\{1-\sqrt{1-\frac{h^2}{\varphi_1^2}\frac{\mu+1}{\mu-1}}\right\}$$

or, developed into a power series of h^2,

$$x_1 = \tfrac{1}{2}\frac{\mu h^2}{\varphi_1(\mu-1)}\left\{1+\tfrac{1}{4}\frac{\mu+1}{\mu-1}\frac{h^2}{\varphi_1^2}+\cdots\right\} .$$

For our purpose it will be enough to retain the first term, thus

$$x_1 = \tfrac{1}{2}\frac{\mu h^2}{\varphi_1(\mu-1)} . \tag{10}$$

This is to be substituted into the equation (9). With the abbreviations

$$c = f_2 - \mu f_1 , \qquad \varphi_1 - D = f_1$$

equations (7) and (9) will become

$$\left.\begin{aligned} &(\sigma_2) && \sqrt{(f_2-x)^2+y^2} - \mu\sqrt{(f_1-x)^2+y^2} = c \\ &(P_1F_1) && x+(\varphi_1-x_1)\frac{y}{h} = f_1 , \end{aligned}\right\} \tag{11}$$

which are two equations for x_2, y_2. Of these only y_2 will be required. In fact, we have

$$\frac{1}{\sin u'} = \frac{l_2}{y_2} = \frac{c}{y_2} + \frac{\mu l_1'}{y_2} = \frac{c}{y_2} + \mu\frac{l_1+l_1'}{h},$$

and since by the original equation of the front surface

$$\mu(l_1+l_1') = \mu\varphi_1 - l_0 = \mu\varphi_1 - x_1,$$

$$\frac{h}{\sin u'} = \frac{ch}{y_2} + \mu\varphi_1 - x_1. \tag{12}$$

Thus it remains to find y_2 from the equations (11).

Now, the first of these can be written, up to y^4-terms and after simple reductions,

$$(\sigma_1) \qquad y^2\left(\frac{\mu}{f_1-x} - \frac{1}{f_2-x}\right) = 2(\mu-1)x,$$

and substituting here $y/h = (f_1-x)/(\varphi_1-x_1)$ from the second of equations (11),

$$\frac{h^2(f_1-x)}{(\varphi_1-x_1)^2}[\mu f_2 - f_1 - (\mu-1)x] = 2(\mu-1)(f_2-x)x,$$

an equation for $x = x_2$. The abscissa x can be developed into a power series of h^2, say

$$x = a_0h^2 + a_2h^4 + \cdots,$$

with coefficients to be determined. For our purpose it is enough to find a_0. Now, up to h^4-terms,

$$\frac{f_1-a_0h^2}{(\varphi_1-x_1)^2}[\mu f_2 - f_1 - (\mu-1)a_0h^2] = 2(\mu-1)(f_2-a_0h^2)(a_0+a_2h^2),$$

where, by (10), x_1 is itself proportional to h^2. Thus, substituting the value of x_1 and comparing the terms free from h, we find a_0 and thence the required abscissa up to h^4,

$$x = x_2 = \frac{f_1(\mu f_2 - f_1)}{2(\mu-1)f_2\varphi_1^2}h^2. \tag{13}$$

The corresponding ordinate will be, by the second of (11),

$$y_2 = \frac{hf_1}{\varphi_1}\left(1 - \frac{x_2}{f_1} + \frac{x_1}{\varphi_1}\right)$$

or, with the required approximation,

$$\frac{h}{y_2} = \frac{\varphi_1}{f_1}\left(1+\frac{x_2}{f_1}-\frac{x_1}{\varphi_1}\right),$$

and by (12),

$$\frac{h}{\sin u'} = \frac{c\varphi_1}{f_1}\left(1+\frac{x_2}{f_1}-\frac{x_1}{\varphi_1}\right)+\varphi_1\left(\mu-\frac{x_1}{\varphi_1}\right),$$

where both x_1/φ_1 and x_2/f_1 are of the order of h^2/φ_1^2.

Ultimately, substituting x_1 and x_2 from (10) and (13), we have, up to terms of the order of h^4/φ_1^4,

$$\frac{h}{\sin u'} = \frac{\varphi_1 f_2}{f_1} - \frac{(1+\mu)f_2-\mu f_1}{2(\mu-1)f_2\varphi_1}\,h^2 . \tag{14}$$

Thus, *the sine-condition*

$$\frac{h}{\sin u'} = \frac{\varphi_1 f_2}{f_1} = \text{const}.$$ [2]

will be satisfied up to h^4-terms if the free parameters of the lens are chosen so as to make $(1+\mu)f_2-\mu f_1=0$, *i.e.*

$$\frac{f_1}{f_2} = \frac{1+\mu}{\mu} . \tag{15}$$

Such a lens will be *aplanatic* up to the fourth power of the relative aperture.

Since $\mu>0$, we have $f_1>f_2$. Thus, if the front surface is convex, so will be the back surface. If the front surface is concave, its focus F_1 is in front of the lens, and much the same reasoning as above will show that in this case only a dispersive aplanatic lens of the type here considered is possible, whose back surface is also concave. In the sequel, however, having in view an object glass, we may confine our attention to the preceding case of a *collective* aplanatic lens, whose two surfaces σ_1, σ_2 are determined by (6), (7), and (15).

4. *Aplanatic Lens Achromatised.*—The aplanatic lens just described, and defined by equations (6), (7), (15), can be achromatised (paraxially at least) by splitting it by means of a spherical interface into two cemented component lenses, L_1 and L_2, having the same refractive index $\mu=\mu_D$ but different dispersions. If, in usual symbols, $\nu=\dfrac{\mu-1}{\Delta\mu}$

[2] The value of this constant is $\varphi_1 f_2/f_1=\varphi_1\varphi_2/(\varphi_1-D)$, which is the focal length of the lens, as it should be.

has the value ν_1 for L_1 and ν_2 for L_2, the condition of achromatism, neglecting the glass thickness, is

$$\frac{1}{\nu_1 F_1}+\frac{1}{\nu_2 F_2}=0\ ,$$

where F_1, F_2 are the focal lengths of the two lenses. If ρ_1, ρ_2 are the curvatures (in absolute value) of the outer surfaces σ_1, σ_2 at their axial points O_1, O_2, and ρ the curvature of the interface, we have

$$\frac{1}{F_1}=(\mu-1)\,(\rho_1\pm\rho)\ ,\quad \frac{1}{F_2}=(\mu-1)\,(\rho_2\mp\rho)\ ,$$

the upper or lower signs to be taken according as the interface is concave or convex towards L_1. If $\nu_1>\nu_2$, the former is the case, and the required curvature of the spherical interface will be

$$\rho=\frac{\rho_1\nu_2+\rho_2\nu_1}{\nu_1-\nu_2}\ . \tag{16}$$

It remains to calculate ρ_1, ρ_2 from the equations (6) and (7). The well-known geometrical formula

$$\rho=|x''(1+x'^2)^{-3/2}|\ ,\ x'=dx/dy\ ,$$

applied to these equations, gives

$$\rho_1=\frac{\mu}{(\mu-1)\varphi_1} \tag{17$_1$}$$

and, taking account of (15),

$$\rho_2=\frac{1}{f_2}\left|\frac{1+\mu-\mu^2}{\mu^2-1}\right|\ . \tag{17$_2$}$$

This completes the set of formulae required for the numerical construction of an achromatic aplanatic lens. In fact, μ, D and φ_1 being chosen arbitrarily, the equations (6), (7), (15) determine the outer surfaces σ_1, σ_2 and, after ν_1, ν_2 have been selected, the last values of ρ_1, ρ_2 substituted in (16) give the curvature of the spherical interface.

As we already know, the meridian curve of the front surface is an ellipse. A few remarks may now be added about the back surface (7), especially in the neighborhood of its axial point. If $x'=dx/dy$, the sign of the abscissa x of a point of the surface near the axis will be that of x_0'', by Taylor's theorem and since $x_0'=0$. This can be shortly written sign $x=$sign x_0''. With this notation equation (7) gives at once

$$\text{sign } x=\text{sign }[\mu^2-\mu-1]\ .$$

Thus if, as assumed, the back surface σ_2 is to be *convex* at the axis, we must have $\mu^2-\mu-1<0$ or $\mu<\mu_0$, where μ_0 is the positive root of

$$\mu^2-\mu-1=0\,,$$

i.e.

$$\mu_0=\frac{1+\sqrt{5}}{2}\doteq 1.6180\,. \tag{18}$$

Such then is the upper limit of μ for an axially convex back surface. (If $\mu=\mu_0$, the surface is flat, and if $\mu>\mu_0$, it is concave at O_2.) This surface, being of the fourth degree, may of course change the sense of its curvature at some finite distance from the axis, *i.e.*, its meridian curve may have there an inflection point. Finally let us recall that the focal length of the whole system is, in accordance with (14), $f=\varphi_1 f_2/f_1$, and for an aplanatic lens, by (15),

$$f=\frac{\mu\varphi_1}{\mu+1}\,. \tag{19}$$

For the sake of illustration of the method a numerical example of such an achromatised aplanatic lens may now be given.

Let

$$\mu=\mu_D=1.6118\,,$$

which corresponds to an actual pair of glasses of Chance Bros., having to four figures the same μ_D and at the same time sufficiently different ν values,

$$\nu_1=59.0\,,\quad \nu_2=36.9\,.$$

Put, in any conventional units, $f_2=100$. Then, by (15),

$$f_1=\varphi_1-D=162.04\,.$$

Further, take for the total thickness[3]

$$D=D_1+D_2=20\,,$$

sufficient to give an available aperture $2h=50$. Then $\varphi_1=182.04$, whence, by (17_1), (17_2),

$$r_1=\frac{1}{\rho_1}=69.099\,,$$

$$\rho_2=0.00008699\,,\quad r_2=11496\,;$$

the back surface will thus be but slightly convex, almost flat, at the axis, which is due to the fact that the chosen μ lies just below the

[3] The correction of (16) for this large thickness need not detain us in this merely illustrative example.

critical value $\mu_0 = 1.618$. Cf. (18). The focal length of the system will now be, by (19),

$$f = \frac{\mu\varphi_1}{\mu+1} = 112.34 ,$$

and therefore, the geometrically available relative aperture almost $f/2$. Next, by (16) and with the quoted ν-values of the Chance glasses,

$$\rho = 0.024396 , \quad i.e. \; r = 40.922 .$$

Finally, to secure a large aperture, the thickness of the first component has to be made as large as possible, say, $D_1 = 18$. Then $D_2 = 2$.

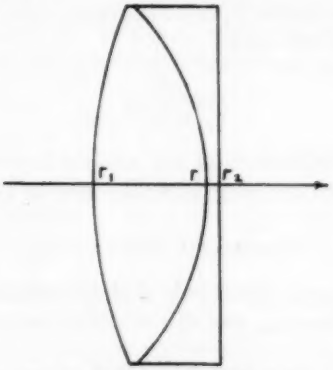

FIG. 6. *Achromatised aplanatic lens.*

This completes all the data of the lens, which is drawn to scale in Fig. 6. In this drawing the slightly curved shape of the back surface could not be shown. (See the next figure.) The lens has an available aperture $2h$ equal to about 50, or, since, $f \doteq 112$, a relative aperture $f/2.2$. This can be increased by making the lens thicker.

In conclusion a few numbers may be quoted which will give a correct idea about the amount of deviation of the aspherical surfaces σ_1 and σ_2 from their osculating spheres constructed at the axial points, and having the radii r_1, r_2 just written down. Let $\Delta x = x - x_0$ be the difference of the abscissae of the meridian curve of the surface and of its curvature circle for equal ordinates y. Then, for the front surface (6), the required deviation turns out to be, up to terms in y,[6]

$$\Delta x = -\frac{\mu\varphi_1}{8(\mu-1)^3}\left(\frac{y}{\varphi_1}\right)^4 , \qquad (\sigma_1)$$

and for the back surface (7), differentiating with respect to $u=y^2$ and writing $x'=dx/du$,

$$\Delta x=\frac{1}{8}[4x_0''-\rho_2^3]y^4\,, \qquad (\sigma_2)$$

where $4(\mu-1)^2x_0''=\frac{1}{\varphi_2^3}(\mu^4-3\mu^3+\mu^2+3\mu+1)$. Substituting the above numerical values of the constants we have for the front surface

$$\Delta x=-0.1458\left(\frac{y}{\varphi_2}\right)^3 y\,. \qquad (\sigma_1)$$

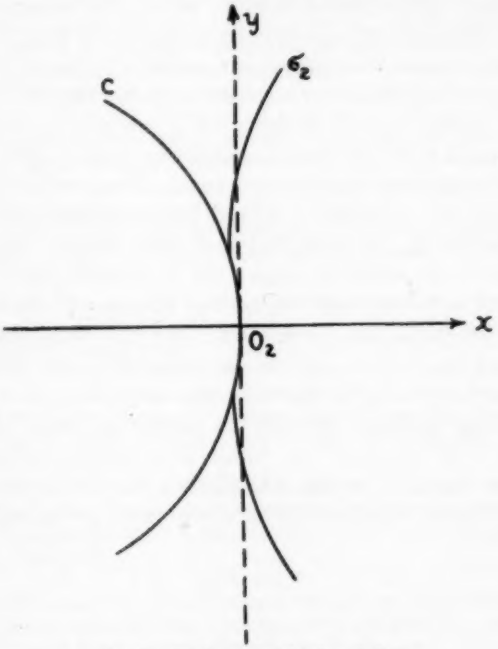

FIG. 7. *Back surface of aplanatic lens.*

Thus, e.g., for $y=25$ (which is just above the extreme aperture of the lens compatible with its thickness), $\Delta x=-0.057$, *i.e.*, for $\varphi_2=100$ mm only about -0.06 mm. For $y=12.5$ the deviation Δx is 16 times smaller, and so on. The negative sign of Δx signifies that the meridian curve of σ_1 lies between the curvature circle and the tangent at the axial point O_1. The middle surface is, as will be remembered, a sphere. For the back surface we have from the data of the lens $4x_0''=1.025\ 10^{-6}$,

always with $\varphi_2 = 100$, while $\rho_2 = 8.7.10^{-6}$ and $\rho_2^3 = 7.10^{-12}$ which is negligible in presence of $4x_0''$. Thus the deviation is reduced to $\Delta x = \frac{1}{4}x_0''$, *i.e.*

$$\Delta x = +0.1281\left(\frac{y}{\varphi_2}\right)^3 y\,. \qquad (\sigma_2)$$

The sign of Δx being positive, the Cartesian surface σ_2 lies outside the osculating sphere and, just near the optical axis at least, between this sphere and the tangential plane at O_2. In a certain distance y from the axis, however, σ_2 crosses this plane ($x = 0$), having first changed the sign of its curvature, as shown in Fig. 7 (where c is the curvature circle). In fact, as follows from a computation which can here be omitted, x vanishes for $y/\varphi_2 = 0.1843$. Thus the meridian curve of σ_2 crosses the y-axis at $y = 18.43$, which corresponds to about the extreme aperture of the lens. Notice that the height $y = h'$ at which a ray impinges upon σ_2 and emerges from the lens is somewhat smaller than the height $y = h$ of the incident ray, parallel to the axis. Their ratio turns out to be, approximately, $h' : h = 0.89$. As was just mentioned, the surface σ_2, being convex at O_2, becomes flat and then slightly concave before reaching $y = 18.43$, where it crosses the tangential plane. It will be kept in mind, however, that this surface has even at the apex O_2 only the very small curvature $\rho_2 = 0.000087$, and this decreases still in the neighborhood of O_2, the surface becoming flat and then concave before it reaches $y = 18.43$. Between this height and $y = 0$ the surface always remains enclosed between the tangential plane and the very large sphere $r_2 \doteqdot 115\ \varphi_2$, so that within these limits it differs but very slightly from a plane. At any rate the last-written formula (σ_2) gives all the information required for shaping this surface with sufficient accuracy.

ROCHESTER, N. Y.
JUNE, 1925.

INSTRUMENT SECTION

METHODS OF VARYING THE SENSITIVITY OF GALVANOMETERS*

By F. Wenner

ABSTRACT

Variation of the sensitivity of galvanometers. Some of the methods for reducing the sensitivity of galvanometers, which have been the subject of recent publications as well as some of the better methods described more than 10 years ago, but not generally known, are discussed, and it is shown that the latter are applicable under conditions not previously considered.

Series or Parallel Resistance. When the quantity being measured must be considered to be proportional to an electromotive force rather than to a current, the sensitivity may be reduced either by putting a resistance into the circuit in series or in parallel with the galvanometer. However, if a series resistance is used the resistance of the circuit, to an electromotive force developed within the galvanometer, is increased, while if a parallel resistance is used this resistance is decreased. Thus, not only is the damping changed, but if the electromotive force is transient or alternating the amount of resistance which must be used to bring about a specified reduction of the sensitivity cannot readily be determined since it depends in a complicated way upon the resistance of the source of the electromotive force and upon the constants of the galvanometer.

Series and Parallel Resistance. By the use of two series resistances and a parallel resistance the resistance of the circuit, considered both from the standpoint of the source of the electromotive force and from the standpoint of the galvanometer, may be kept constant while the sensitivity is reduced as much as may be desired. However, the damping may be kept constant and the sensitivity reduced in any desired ratio by means of a single series resistance and a parallel resistance. The parallel resistance may be connected either directly across the galvanometer or across the galvanometer and the series resistance. For the former arrangement

$$S=(n-1)X \text{ and } P=nX/(n-1)$$

where X is the resistance of the source of the electromotive force, S is the series resistance, n the ratio of the normal sensitivity to the sensitivity desired, and P the parallel resistance. For the latter arrangement

$$T=(n-1)X/n \text{ and } P=X/(n-1)$$

where T is the series resistance. These formulas are independent of the type and the constants of the galvanometer used, and with limitations which are considered apply whether the electromotive force is direct, impulsive, alternating, or transient.

INTRODUCTION

In the use of galvanometers it is sometimes desirable or necessary to reduce the sensitivity, otherwise the deflection will be beyond the limits of the scale. If the magnitude of the quantity being measured is determined by means of the deflection of the galvanometer, to avoid a

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redetermination of the constant for each sensitivity used, it is necessary that the sensitivity be reduced by a specified factor. Where the quantity being measured is proportional to the current, whether direct, impulsive, alternating, or transient, the well known method proposed by Ayrton and Mather[1] meets the ordinary requirements in a fairly satisfactory manner.

However, when as is more frequently the case, the quantity being measured must be considered to be proportional to an electromotive force rather than to a current, the matter is more complicated, and satisfactory methods by which the sensitivity may be reduced in any desired ratio do not seem to be generally known. Therefore, a fairly satisfactory method proposed by Volkmann[2] and two simpler and more satisfactory methods will be described. These latter have been known, at least to some extent, for ten years or more, though for the most part only casual reference has been made to them in publications,[3] and the generality of their applicability has not been fully pointed out. Also other methods which recently have been the subject of publications will be considered. This paper will, therefore, be more in the nature of a brief discussion of those phases of the subject which seem not to be generally understood than a presentation of anything new.

THE SERIES AND THE PARALLEL RESISTANCE METHODS

When the quantity being measured is proportional to an electromotive force the current through the galvanometer and consequently the deflection or the sensitivity can be reduced by putting a resistance into the circuit in series. Also since both the galvanometer and the source of the electromotive force have resistance, the sensitivity can be reduced by putting a resistance in parallel with the galvanometer. In either case it is a simple matter to determine what resistance should be put in series or what resistance should be put in parallel to reduce the sensitivity to a direct electromotive force to any desired fraction of the initial or normal sensitivity. For example, if the source of the electromotive force has a resistance X and the galvanometer has a resistance G, and it is desired to reduce the sensitivity to $1/n$ normal by a series resistance, this resistance must be made equal to

$$(n-1)\ (X+G).$$

[1] Ayrton and Mather, Electrician *32*, p. 627; 1893.

[2] Volkmann, Ann. der Physik *10*, p. 217; 1903.

[3] Wenner, Bureau of Standards Bull. *8*, p. 590; 1912; (Reprint No. 181); and Scientific Paper 273, p. 223; 1916.

White, Zeits. f. Inst. *34*, p. 80; 1914.

Laws, Electrical Measurements, p. 189, 1917.

However, when a series resistance is used for reducing the senistivity, the resistance to the electromotive force developed by the motion of the moving system of the galvanometer is increased. Therefore, in case the galvanometer is of the modern type without a damping coil and the motion of the moving system is properly damped initially, it will not be sufficiently damped with the series resistance. If it is desired to reduce the sensitivity by the same factor by means of a parallel resistance this resistance must be made equal to

$$GX/(n-1)\ (G+X).$$

In this case the resistance to the electromotive force developed by the motion of the moving system of the galvanometer is decreased. Therefore, in case the motion is properly damped initially, it will be overdamped with the parallel resistance. Consequently, except for a very limited range of sensitivities, there is either practically no damping or an excessive overdamping, and neither condition is conducive to rapid and accurate work nor to the peace of mind of the person making the observations.

The use of a series resistance for reducing the sensitivity to impulsive electromotive forces, that is, for reducing the sensitivity of galvanometers used ballistically, has been considered rather fully by Masius[4] who has also considered the use of a parallel resistance for the same purpose. For either method the relations between the various constants of the galvanometer and of the circuit are rather complicated, certain constants of the galvanometer must be determined, and then the computations are tedious unless use is made of tables, which so far have been published only in part, or of Randall's universal calibration curve.[5] Further, in general only galvanometers which have small damping on open circuit have been considered, and finally there are the same troubles on account of under or over damping as when the electromotive force is direct.

When either of these methods is used and the electromotive force is alternating or transient the amount of resistance which must be used to bring about a specified reduction of the sensitivity depends in a complicated way upon the type and constants of the galvanometer, as well as upon the constants of the source of the electromotive force. Consequently an analysis of the problem would lead to more detail than would be in keeping with the purposes of this paper.

[4] Phys. Rev. *23*, p. 649; 1924; and *25*, p. 211; 1924.
[5] Phys. Rev. *25*; p. 86; 1925.

The situation with respect to the use of either a series or a parallel resistance for reducing the sensitivity of galvanometers when the deflection must be considered as proportional to an electromotive force rather than to a current, therefore may be summarized as follows:

(a) If the object sought is merely a protection of the galvanometer while an approximate balance of a bridge, potentiometer, or other device is being made, and final adjustment is to be made with normal sensitivity, either method is fairly satisfactory.

(b) If the object sought is a reduction of the sensitivity to a specified fraction of the normal sensitivity so that deflections may be read and

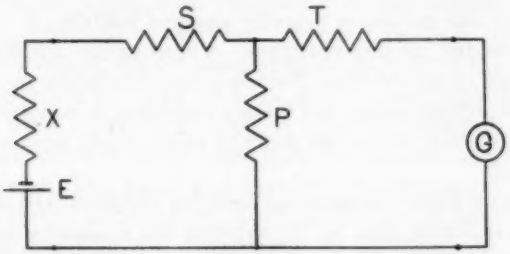

FIG. 1. *Arrangement used by Volkmann for reducing the sensitivity by a specified factor while keeping the damping constant.*

recorded, neither method is satisfactory, whether the electromotive force is direct, impulsive, alternating or transient, except for very small reductions in the sensitivity.

THE VOLKMANN METHOD

The first successful attempt to change the sensitivity and at the same time keep the damping constant seems to have been made by Volkmann[6] who used two series and one parallel resistance, connected as shown in Fig. 1. Here X represents the resistance of the apparatus in which the electromotive force, which is to be measured by the deflection of the galvanometer, is developed; G represents the resistance of the galvanometer; and S, P, and T represent the three added resistances. These three resistances have values so chosen as to satisfy the following conditions:

1. The sensitivity is any desired fraction of the normal sensitivity, that is, the sensitivity with both S and T equal to zero and P equal to infinity.

[6] Ann. der Physik, *10*, p. 217; 1903.

2. The resistance to the electromotive force developed by the motion of the coil of the galvanometer is equal to $X+G$.

3. The resistance to the electromotive force to be measured is equal to $X+G$.

Later Edler[7] made an independent derivation of the relations between the various resistances and obtained equations from which the values for S, T, and P may be determined much more readily than from the equations given by Volkmann. Further, he showed that values of S, T, and P which satisfy these three conditions when the electromotive force is direct also apply when the galvanometer is used ballistically, that is, they apply when the electromotive force is impulsive.

However, many of those who have had extensive experience in the use of galvanometers no doubt have noticed that usually critical damping may be brought about by the use of suitable resistance connected into the circuit in series or a suitable resistance connected in parallel with the galvanometer depending upon whether the resistance of the source of the electromotive force being measured is less or greater than the external critical resistance of the galvanometer. Also that when the parallel resistance is used the sensitivity is less than normal. Obviously if the resistance of the source of the electromotive force is equal to the external critical resistance of the galvanometer it may in effect be made greater simply by putting a suitable resistance in series. Then by putting a suitable resistance in parallel with the galvanometer the sensitivity is made less than normal and the damping is made critical as it was without these two resistances. Thus, a reduction of the sensitivity is brought about by the use of a single series and a parallel resistance, and at the same time the damping is kept constant. Obviously, therefore, the method proposed by Volkmann is unnecessarily complicated.

METHODS A AND B

Experience shows that, in general, there is no reason why the resistance to the electromotive force being measured should be kept constant. Consequently, Volkmann's condition (3) may be considered to be superfluous. We then find that his conditions (1) and (2) may be satisfied either without the resistance S or without the resistance T. If the resistance T is omitted, the sensitivity may be reduced to $1/n$ normal by making

$$S=(n-1)X \text{ and } P=nX/(n-1).$$

[7] Elek. u. Masch-b. *33*, p. 165; 1915.

This will be referred to as method A. If the resistance S is omitted the sensitivity may be reduced to $1/n$ normal by making

$$P=X/(n-1) \text{ and } T-(n=1)x/n$$

This will be referred to as method B.

It will be noticed that neither the resistance or any other constant of the galvanometer is contained in any of these equations. Therefore, values S and P, or P and T which reduce the sensitivity of one galvanometer to $1/n$ normal will reduce the sensitivity of any other galvanometer by the same proportional amount. Further, they reduce the sensitivity by the same proportional amount whether the electromotive force is direct, impulsive, alternating or transient. However, if the apparatus in which the alternating or transient electromotive force is developed is inductive, it is necessary that S and P, or P and T be inductive and that each have the same time constant as X.

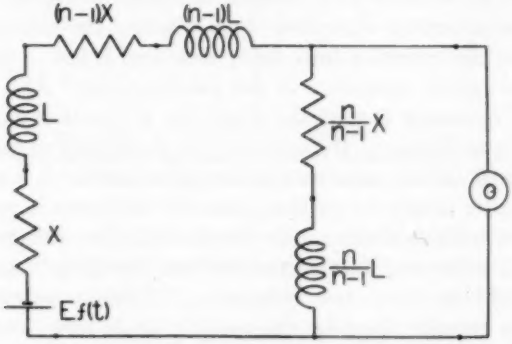

FIG. 2. *Diagram showing in part the correctness of the conclusions stated above with reference to Method A.*

The validity of these conclusions may be shown in different ways. One of these, which is both general and simple, is based upon the following principles:

1. The effect of an electromotive force in one branch of a system of conductors upon the current in another branch is the same as that of an electromotive force introduced into the latter branch equal to the potential difference which would appear across the break were this branch opened. A more complete statement and a discussion of this principle will be found in a Scientific Paper of the Bureau of Standards to be issued shortly.

2. If the resistance and time constant of the entire galvanometer circuit is kept constant while either of two electromotive forces is in-

troduced into the circuit, one of which is $Ef(t)$ and the other of which is $Ef(t)/n$, the deflection of the galvanometer for the latter is at every instant $1/n$ of what it would be for the former. This principle is based on the suppositions that the torque tending to produce a deflection is proportional to the current in the winding, that the retarding torque is proportional to the rate of displacement, and that the impedance is independent of the deflection.

Referring now to Fig. 2 for method A and to Fig. 3 for method B it will be seen that if the galvanometer circuit is open the potential drop across the break is $Ef(t)/n$ and if the galvanometer circuit is closed the resistance and time constant of the entire galvanometer circuit (con-

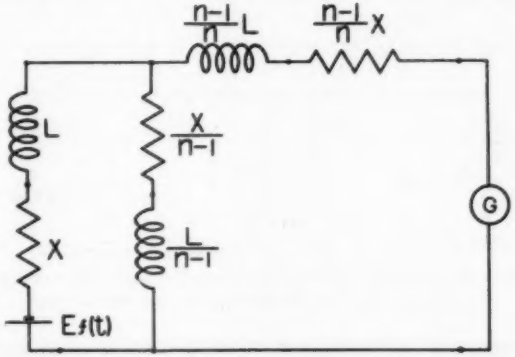

FIG. 3. *Diagram showing in part the correctness of the conclusions stated above with reference to Method B.*

sidered from the standpoint of the galvanometer) is the same as it would be without S and P, or P and T, that is with S equal to zero and P equal to infinity or P equal to infinity and T equal to zero. Thus the factor by which the sensitivity is reduced is entirely independent of the way in which the measured electromotive force varies with time and entirely independent of the constants of the galvanometer.

It should be noted that the time constant of the source may be zero or may result from an inductance in series with a resistance, as shown in Figs. 2 and 3, from a capacitance in parallel with a resistance; or from any combination of inductances, capacitances, and resistances. Obviously, the time constant of the source of the electromotive force does not affect the sensitivity either when the electromotive force is direct, or impulsive, providing that for the latter the time constants of X, S, and P; or X, P, and T differ among themselves only be amounts

which are small in comparison with the damped period of the galvanometer.

It should be pointed out that theoretically there are no limitations to the value which may be taken for n except that it must be positive and equal to or greater than unity. However, if n is taken only slightly greater than unity, P will be excessively large for either method. For this case the sensitivity should be changed by the series resistance method. Further, if n is large it may not be advisable to use method A as this method requires an excessively large value for S. For this case method B should be used unless there is a reason why the current drawn from the source of the electromotive force should be limited. Aside from these special cases there will be but little choice between the two methods.

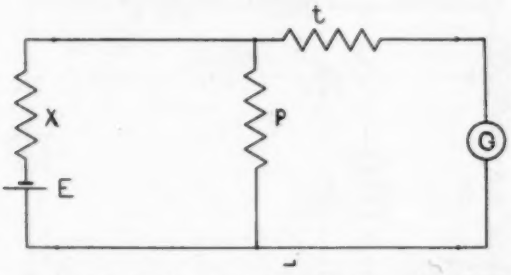

FIG. 4. *Arrangement for bringing about a desired damping and a desired sensitivity.*

It might seem that both of these methods fail in case the resistance of the source of the electromotive force is very small. However, if a resistance which is small in comparison with the resistance of the circuit is put into the circuit in series, it will not materially affect either the damping or the sensitivity. Therefore, such a resistance may be connected to the source of the electromotive force[9] and considered as a part of X. When this is done either method may be used. This, in effect, is the use of these methods in conjunction with the series resistance method, but the reduction in sensitivity brought about by the series resistance is so small that it may be neglected or readily estimated to the required accuracy.

A special feature of these methods is the ease with which they may be applied two or more times simultaneously, used in conjunction with each other, or in conjunction with other methods. For example, the

[9] If the added resistance is considered to constitute all of X method A becomes identical with the method proposed by Blondel. A. Blondel, Ann. d. Physique, *10*, p. 219; 1918.

sensitivity may be adjusted to a desired value either by a series resistance t or a parallel resistance p, or by both a series resistance t and a parallel resistance p, connected as shown in Fig. 4.

When the two resistances are used, both the damping and the sensitivity may be adjusted as desired within limits depending upon the galvanometer. Then the sensitivity may be reduced to $1/n$ of this value by the use of an additional series resistance T and an additional parallel resistance P as shown in Fig. 5.

Further, the sensitivity may be reduced to $1/n'$ times this new sensitivity by the use of an additional series resistance T' and an additional parallel resistance P' as shown in Fig. 6. With plugs for short

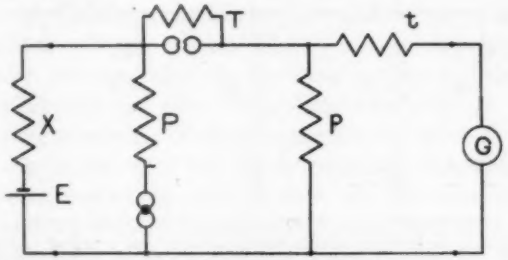

FIG. 5. *Arrangement for bringing about a desired damping and a desired sensitivity, and then reducing the sensitivity by a factor 1/n.*

circuiting either or both T and T' or connecting in either or both P and P' the sensitivity may be instantly changed from one to another of the following values:

1	times the adjusted sensitivity
$1/n$	times the adjusted sensitivity
$1/n'$	times the adjusted sensitivity
$1/nn'$	times the adjusted sensitivity

Similar results may be accomplished by the use of method A or by a combination of the two methods.

While a consideration of the design and construction of "universal shunts" or "sensitivity keys" for galvanometers, or of specific applications of these methods would lead to detailed descriptions which would not be in keeping with the purpose of the paper, nevertheless, the following suggestion as to the procedure to be followed in the use of these methods may not be out of place.

a. Usually it will be advisable to make the apparatus in which the electromotive force to be measured has its source, have a resistance of

an integral number of ohms, and preferably a decimal multiple of 1, 2, or 5 ohms. If this cannot be accomplished in the construction of the apparatus, resistance may be added externally and considered as though it were a part of the apparatus.

b. Parallel or series resistances used to bring about a suitable damping or a suitable damping and a suitable initial or adjusted sensitivity should be kept independent of those resistances used to change the sensitivity by a specified factor. An exception to this rule may or should be made in cases in which it will be necessary to change the sensitivity in both directions from the adjusted value.

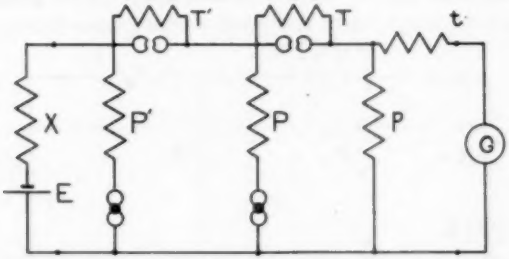

FIG. 6. *Arrangement for bringing about a desired damping and a desired sensitivity, and then reducing the sensitivity by a factor $1/n$, $1/n'$, or $1/nn'$.*

c. Where it will be necessary to reduce the sensitivity by a considerable number of factors, for which n is not an integer but has values such, for example, as 1.32, 1.87, 2.40, etc., it will be advisable to plot curves from which the values of S and P, or T and P may be read for any factor within the range to be used.

While these methods can be applied in almost all cases, the possibility of using other means for reducing the deflection of the galvanometer should not be lost sight of. One of those applicable in various kinds of measurements is a reduction of the electromotive force which is being measured. Where the electromotive force can be reduced readily by any amount and in any desired ratio, this procedure usually will be more satisfactory than a reduction of the sensitivity of the galvanometer. Another means which is often overlooked is that in many cases a less sensitive galvanometer might be used, and that thereby not only would the necessity for reducing the sensitivity be avoided but a more satisfactory performance be secured.

BUREAU OF STANDARDS,
WASHINGTON, D. C.,
MAY 15, 1925.

DIMENSIONAL VARIATION OF DUPLITIZED X-RAY FILM

By J. M. Cork

Mensurational characteristics of photographic films have been reported upon by Cheshire,[1] Curtis,[2] Ross[3] and Davidson.[4] None of these measurements appear to include duplitized x-ray film. As this film is now used extensively in connection with x-ray diffraction apparatus, it becomes necessary to know something regarding its reliability, if accurate determinations of lattice constants and their changes due to various factors, are to be made. In certain types of work it is, of course, possible to photograph simultaneously with the diffraction pattern of the unknown sample, the pattern of a standard substance such as rock salt, in which case only relative measurements are made upon the film and only local distortions will give trouble. However, very often this procedure is not possible, and one must rely upon the constancy of the film, making it desirable to know what factors might influence its dimensions.

In obtaining a number of diffraction patterns to show the effect of some variable factor upon the lattice constant, successive exposures might be taken over a long period of time. It is of importance to know whether it is more desirable to measure all of the films at the same time, or to measure each at some definite interval after its development. Some of the factors whose change might result in a change in the dimensions of the film are humidity, temperature, treatment during development and washing and manner of drying.

To test these effects a series of fine parallel lines about 5 cm apart were placed upon strips of film 5 cm by 30 cm. The lines were made by lightly etching the emulsion on one side with a sharp instrument. Accurate measurements were then made of the distances between lines and the films were then developed and washed. Being exposed to light during the first measurement they became opaque upon development. The films were next measured while wet and then immediately after drying and at regular intervals thereafter. All measurements were made upon a Gaertner comparator reading directly to .001 mm.

RESULTS

Fig. 1 shows a summary of the results obtained. In each case the results are expressed in terms of the average per cent variation from

[1] Optician and Scientific Instrument Maker, *65*, p. 288; 1923.
[2] J.O.S.A. & R.S.I. *7*, p. 275; 1923.
[3] Astrophysical Jour, pp. 59, 181, 1924.
[4] Optical Society Transactions *24*, pp. 1, 41, 1923.

the values obtained upon the film immediately before developing. Curve A shows the average values for three films which were developed in the usual manner and washed for 10 hours. An increase in length of about .5 per cent while wet, and a contraction of about .6 per cent upon

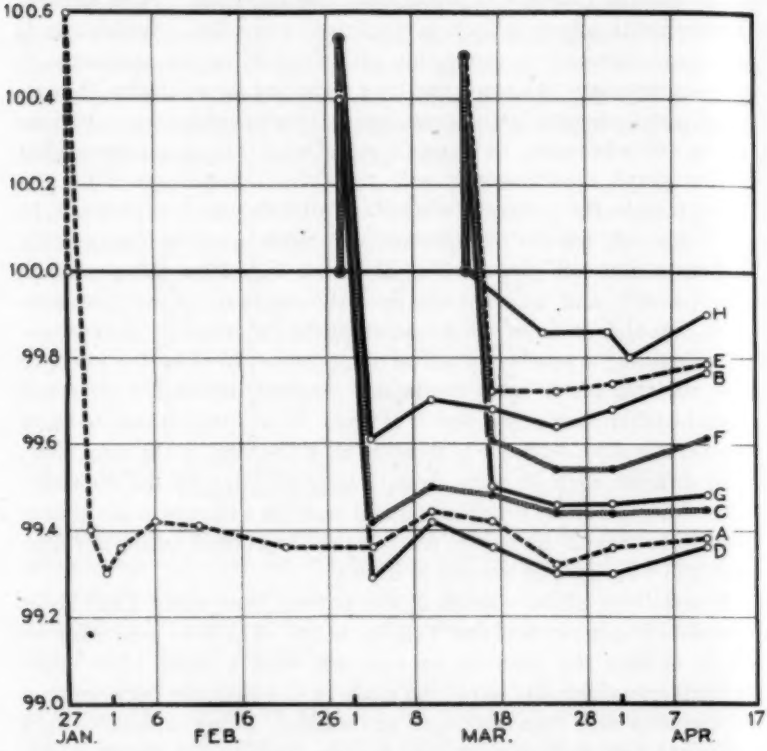

FIG. 1. A Developed, washed 10 hrs.; B Developed, washed 3 hrs.; C Developed, washed 30 hrs.; D Not developed, washed 20 hrs., E Developed, washed 1 hr.; F Developed, washed 7 hrs.; G Developed, washed 30 hrs.; H Not developed, not washed.

drying is observed. Measurements at successive times show fluctuations in this value of about .15 per cent. One month later another group of films was similarly treated except in the period of washing. Curve B shows the average values for those films developed and washed a period of 3 hours; curve C that when the washing period was 30 hours. It appears that the longer time of washing results in a greater contraction in the film. In curve D, the films were neither developed nor fixed but

were washed 20 hours. In this case, the contraction was greater than in the developed films. Another group of films was started two weeks later. The effect of the greater contraction due to the longer washing period is evident in comparing curves E, F and G which were for 1,7 and 30 hours respectively. Curve H shows the fluctuations in a film neither developed nor washed. Variations appear to be as large as .2 per cent in this case.

It is observable that regardless of the time the films were developed the fluctuations at any date were of the same kind. This fluctuation coordinated approximately with the humidity of the atmosphere. The temperature variation was in no case as large as 5°C. However, to test the effects of temperature changes, an attempt was made to measure the coefficient of linear expansion of the film. Here one could obtain either a positive value or an apparent negative value, depending upon the starting conditions and the time of heating, etc. This showed that the humidity factor is perhaps the most important item in producing diurnal variations. It may be observed that while curves E, F and G fit very well with B and C as regards time of washing, curve A lies below C and G, although the time of washing was less, This is probably due to different humidity conditions existing when the pre-development measurements were made.

It is apparent then that if diffraction patterns taken over successive intervals of time are to be compared, care should be taken to give the films identical treatment in washing, developing and drying. Further, the ideal conditions that before and during all exposures the temperature and humidity have the same constant values, are necessary; otherwise the change in pattern due to a change in lattice constant of .1 per cent is apt to be completely masked by variations in the film. While the values given are average values, it very often happened that local distortions were such that the contractions at different places in the same film differed by from .1 to .2 per cent. Thus, even when a comparison pattern is used, these local distortions are of such an order as to give serious difficulty. The only recourse for certain types of work seems to be the use of photographic plates.

UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICHIGAN,
APRIL 15, 1925.

A simple lecture-table Galvanometer of high Sensitivity.—In the recently completed Wakefield Science Hall of Park College, we have devised a method of converting a small reflecting student-galvanometer to a lecture-table galvanometer of high sensitivity which has proven very satisfactory. The mounting is permanent, and only the scale is visible from the lecture room. Deflections are easily read from all the one hundred seats of the room, with window shades in their normal positions.

The scheme is as follows: an opening, twelve feet long by eighteen inches high, was left in the middle of front wall of lecture room about two feet above the blackboard. This opening contains the translucent glass scale. The galvanometer with a shop-made adjustable support for electric light is mounted on the opposite wall of the adjacent room in front of the lecture room. (In our case this room is an apparatus room so that its appearance here is not objectionable.) A small lens between light and galvanometer focuses the reflected beam on the translucent scale, which in Wakefield Hall is at a distance of about thirteen feet. At this scale-distance, a single loop of wire eighteen inches in diameter rotated in the earth's magnetic field produces an easily observed deflection.

The galvanometer terminals and the switch for galvanometer light are at the lecture table. The light used is a G. E. 75 watt P. S. 22 lamp, which when mounted horizontally has its filament in a vertical plane, giving the effect of a straight filament of high luminosity.

R. L. EDWARDS, Park College, Parkville, Mo.

Distribution of alternating Current in irregularly shaped Conductors.—Experiments conducted with probes inserted into longitudinal slits in a wire of elliptical cross-section (major axis 4 cm, minor axis 0.5 cm) show that when it is traversed by a longitudinal 500 cycle A.C., both the amplitude and the phase of the current density traversing a given point of the cross-section depend on the co-ordinates of the point, and it is not possible to draw a curve upon the cross-section such that amplitude and phase are the same for every point upon it. The data are presented in the form of a graph. Other data (not reproduced) were obtained with a conductor of rectangular cross-section. In order to simulate the case of a cable lying on the floor of the ocean and earthed at both ends, such as is used for guiding ships into port, the authors attached an insulated copper wire to the bottom of a glass dish containing mercury; the dimensions were so chosen that the distribution of A.C. in the mercury would resemble that in a possible case of a cable immersed in sea-water, which is said to require that the diameter of the cable be greater than that of the wire in the ratio of the square root of the conductivities of mercury and sea water. For this case the amplitude and phase of the current at various points in the liquid were measured and graphed.—[M. Reich and F. Fischer, Göttingen; ZS. f. Phys. *32*, pp. 327-332; 1925.] KARL K. DARROW

A NEW LABORATORY FOR PRECISION X-RAY RESEARCH

By Alice H. Armstrong and W. W. Stifler

Recent articles in this journal by Hudson[1] and by Allison and Clark[2] contain descriptions of the equipment in use in the x-ray laboratories of Professor William Duane at the Jefferson Physical Laboratory. Since the publication of these papers, a new installation for x-ray investigations has been fitted up under Professor Duane's supervision in the Cruft High Tension Laboratory. The purpose of this article is to describe some of the improved devices incorporated in this new laboratory. The equipment will be discussed under three main headings; namely, high voltage power plant, spectrometer and accessories, and auxiliary apparatus.

High Voltage Power-plant.—Fig. 1 shows one of the four large units, or piers, of the high voltage storage battery which serves as the source of high potential. Such details of the battery as cells, plates, and trays, were planned some time ago by Dr. Howard Blackwell. Professor E. L. Chaffee of the Department of Physics designed the piers and the switch system, and also supervised the construction of the battery. The individual cells are approximately 12 cm high and 5 cm in diameter. The plates are of the Planté type, the active material occupying approximately 5.5 cm by 2 cm by 1 cm on each plate. The plates were supplied by the Willard Storage Battery Company, Cleveland, Ohio, and the jars by the Whitall, Tatum Company, Boston, Massachusetts.

Forty-eight cells, joined in series, contained in an asbestos-board tray covered with several coats of special acid-proof paint, form the unit out of which the battery is built up. One hundred and twenty-eight such trays, arranged on eight shelves, make up the larger unit shown in Fig. 1. The whole battery, as at present assembled, consists of 24,576 cells and gives something over 50,000 volts when connected in series. Four more piers are now being added so that eventually over 100,000 volts will be available.

With such a battery, insulation is a very serious problem. Fig. 1 shows the way in which this has been met. Each of the four piers consists of a series of brick uprights supported on large porcelain insulators

[1] This Journal, *9*, 3, p. 259; 1924.
[2] This Journal, *8*, 5, p. 681; 1924.

set into the concrete floor. These uprights support the shelves for the trays. Between every two shelves there is inserted in the brick uprights a layer of sulphur. The wires connecting one pier to the next are carried on large insulators shown at the top of the picture. To avoid corona loss, all connections are made with bare copper wire approximately 7 mm. in diameter.

FIG. 1. *High voltage storage battery.*

The switches for connecting the groups of trays in series or in parallel are bolted to beams on the front of each pier. These switches are manipulated by an eight foot pole. The exact dimensions of the piers were dictated by the space available in the basement of the Cruft Laboratory for housing the battery. Each pier is approximately 2.4 m high, 1.4 m wide, and 3.2 m deep. The whole battery occupies a floor space approximately 10 m by 3.5 m.

The charging unit is shown in Fig. 2. It consists of a three-phase 550-volt induction motor direct-connected to four 550-volt direct current generators which are connected in series. Each generator is

imbedded in a sulphur base block. For charging, the battery is connected in thirty-two parallel sections, each section consisting of sixteen trays in series. The usual charging current is 2 amperes at approximately 1800 volts. A reverse current relay acts as a circuit breaker and prevents a back current from the battery in case the generator slows down. An over-load relay acts as a circuit breaker if the current becomes too high.

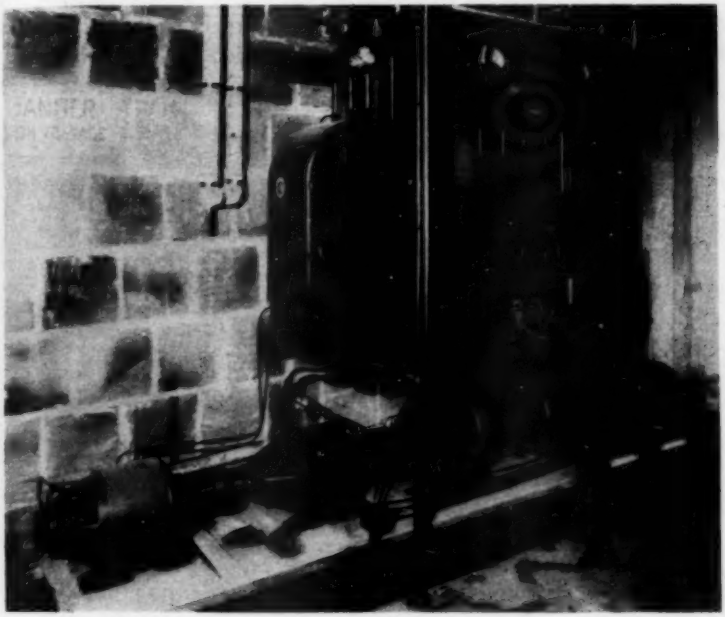

FIG. 2. *Charging unit for high voltage battery.*

The two terminals of the battery are connected to the central posts of two highly insulated, single pole, double throw, lock switches which are fastened to the ceiling. These switches are the regular commercial type, rated for 25,000 volts. Only one of the end terminals on each switch is used at present. The leads to the x-ray laboratory on the floor above are carried through the floor. Fig. 3 shows a temporary arrangement with only the negative lead highly insulated where it goes through the ceiling. This is made possible by the fact that, at present, the negative terminal is connected to the x-ray tube while the positive terminal of the battery is connected to earth through a water

rheostat. Both terminals of the battery are protected by fuses of fine wire, inserted at *A*, Fig. 3. Flexible stranded wire is attached at *B* so that in case a fuse blows, the loop, *C*, will fall to a vertical position. The shape of the connection to each switch is designed to prevent an arc from running up the wire and striking to the ceiling.

The battery gives a perfectly steady current as high as forty milliamperes for two or three hours with no greater fall in terminal voltage than can be compensated by adjusting the water rheostat, mentioned above. This steadiness not only contributes largely to the accuracy of the work but also increases the speed with which observations can be made, since little time is lost in making adjustments of the current

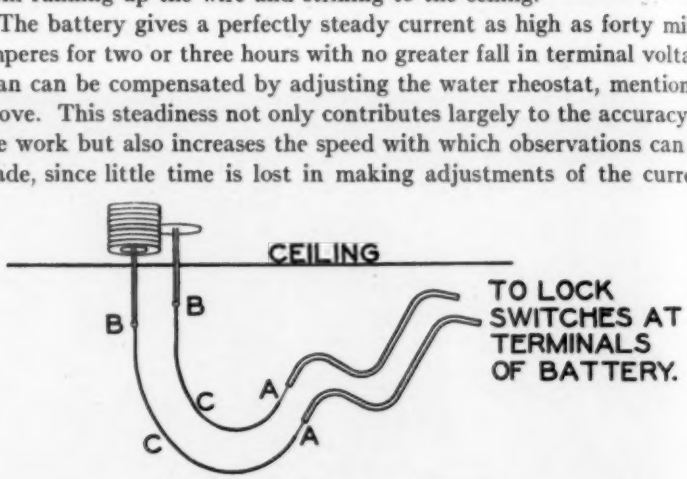

FIG. 3. *Arrangement for fuses.*

through the tube. The successful performance of the battery is due in no small degree to the painstaking care of Mr. C. Lanza who did much of the work of assembling it and who is now responsible for its care.

Spectrometer and Accessories.—The spectrometer is a large Société Genevôise instrument, originally intended for optical work. It has been rebuilt to adapt it to x-ray spectroscopy. The table is equipped with two scales, each graduated to five minutes of arc. On one of these scales the positions of crystal table and ionization chamber can be set and read directly with an error not greater than one minute of arc. By means of two measuring microscopes, one attached to the crystal table mounting and the other to the ionization chamber arm, readings may be taken on the other scale. The filar micrometer eye-pieces on these microscopes give readings directly to one second of arc. Settings of the crystal table, accurate within two seconds of arc, can be made.

The ionization chamber of pyrex glass is the standard form described in the references given above. It is well protected by a heavy lead jacket. The weight of this jacket makes necessary a counter-weight on the other side of the spectrometer.

The spectrometer is mounted on a heavy cast-iron base which in turn is supported by a stout table resting on the concrete floor. The iron base carries two uprights which support a shelf slightly above the level of the crystal table. A remodelled Dolezalek electrometer is mounted on this shelf. The electrometer needle is hung by a suspension 35 cm long, half phosphor bronze and half silver ribbon. With the scale at 2 m, the sensitiveness is approximately 3 meters per volt on the quadrants, the exact value depending on the potential applied to the needle. A cylindrical shield of sheet lead about 6 mm thick protects the electrometer from any stray radiation.

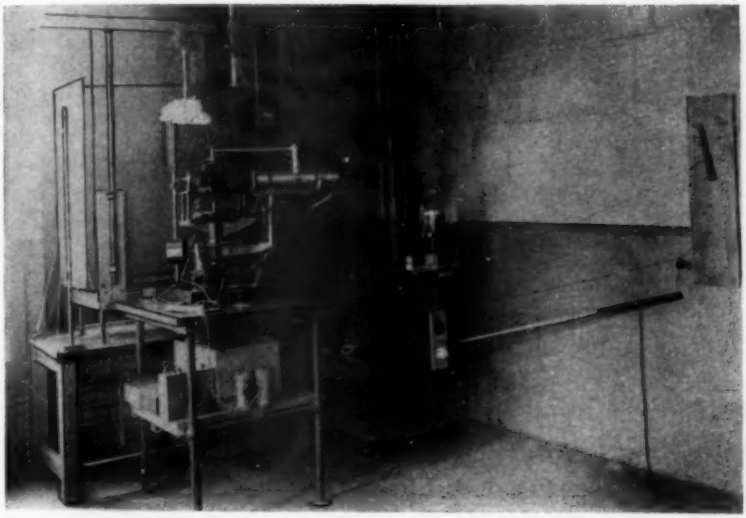

FIG. 4. *Spectrometer, insulated filament control stand, and housing for x-ray tube.*

The lead-covered box which houses the x-ray tube is placed directly behind the spectrometer table. Fig. 4 shows the general arrangement of the spectrometer and accessories. A sketch showing a horizontal section of the apparatus appeared in a recent article in the *Proceedings of the National Academy*.[3]

Auxiliary Apparatus.—The necessary auxiliary apparatus, such as insulating stand for filament battery, control rheostats, etc., etc., are of the general type described in some detail by Allison and Clark

[3] Proc. Nat. Acad. Sci., *10*, 9, p. 374; 1924.

(l.c.). The chief improvement introduced in this arrangement is in the design of the wheel rheostat used as the fine adjustment in regulating the current through the filament. In the new design, the rim of the wheel is grooved so as to leave two ridges several millimeters apart. On each of these ridges a piece of german silver resistance wire is wound. One end of each wire is insulated from the other wire. The other ends of the two wires are connected together. The current enters one of the wires through a suitable sliding contact and leaves the other wire by a second contact. The amount of wire in the circuit is adjusted by revolving the wheel. Mercury contacts to the wires were found to be thoroughly unsatisfactory and a source of constant trouble, but spring contacts of heavy phosphor bronze strip have proved very reliable. This arrangement of a double wire avoids the annoyance of a flexible connection to the wheel which is necessary with a single wire. To increase the delicacy of this adjustment, a resistance of approximately one ohm is shunted across the wheel. The coarse adjustment of the filament current is made through a tube rheostat with sliding contact of the ordinary type. The contact on this rheostat is adjusted by a long wooden pole equipped with a ground connection at the handle as an added protection. The wheel rheostat, switches, etc., on the insulating stand are operated from a distance of about two meters. Enameled silk fish line is used for these string controls.

The milliameter has been provided with a shunt by which its range can be changed as required. When in use it is normally short-circuited to protect it, in the way described by Allison and Clark.

The voltage across the tube is read by an electrostatic voltmeter built by Professor Chaffee and described by D. L. Webster.[4] This instrument rests on a shelf supported from the ceiling and is read by observing the deflection of a spot of light on a large curved scale about 3 meters distant. This scale is approximately 3½ m long and is divided into 250 divisions. For 50,000 volts the deflection is about 220 divisions, and voltages of this magnitude can be read within 100 to 150 volts. The voltmeter is shown at the left, *D*, and the end of the scale at the right, *H*, in Fig. 5.

The connections from the battery switch to the voltmeter and the tube are made with stranded wire heavily insulated with rubber (known commercially as C9 wire). While the insulation cannot be depended upon as a sure protection at high voltages, it aids in reducing brush discharge.

[4] Phys. Rev. 7, p. 599; 1916.

The successful operation of this installation is due in no small degree to the care taken in arranging the connections from the battery to the auxiliary equipment and to the tube. The leads from the basement go directly to two large single throw, single pole lock switches (commercial type). These switches, like those in the basement and on the battery piers, were supplied by the Westinghouse Electric & Manufacturing Company. The switch at the positive end is connected directly to the electrodes of two variable water rheostats connected in parallel. The other electrodes of these rheostats are connected to earth. The positive terminal of the x-ray tube and the moving system of the voltmeter are connected to earth also.

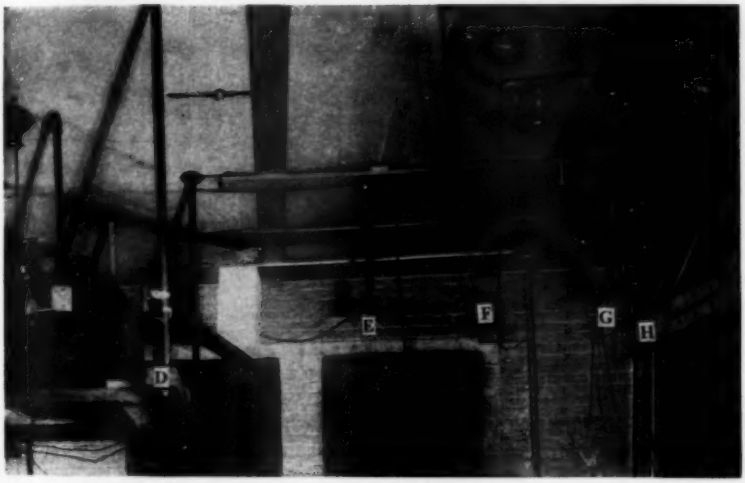

FIG. 5. *Voltmeter and switch system.*

From the switch at the negative terminal, a wire is carried directly across the room at a height of about 3 m. This wire is stretched taut between suitable strain insulators bolted to the walls. It is also supported from the ceiling by enameled silk cord. To the end of this wire is soldered a metal rod which ends in a sphere about 6 cm in diameter. This rod is bent back in a vertical plane so that there is a length of about 50 cm (terminating in the sphere) about 15 cm below the main wire and parallel to it. This is shown at *F* in Fig. 5.

The connection from the voltmeter is supported on porcelain insulators on the ends of large dowel rods about 50 cm long. Though these

insulators are rated for use at 2000 volts, the wooden supports give enough additional insulation to make them quite satisfactory. The wire from the voltmeter is soldered to a brass rod, a few mm in diameter and about 70 cm long, which ends in a sphere about 3 cm in diameter. The joint between wire and rod is wrapped with insulating tape in such a way that it is flexible and yet does not bend so sharply as to break the

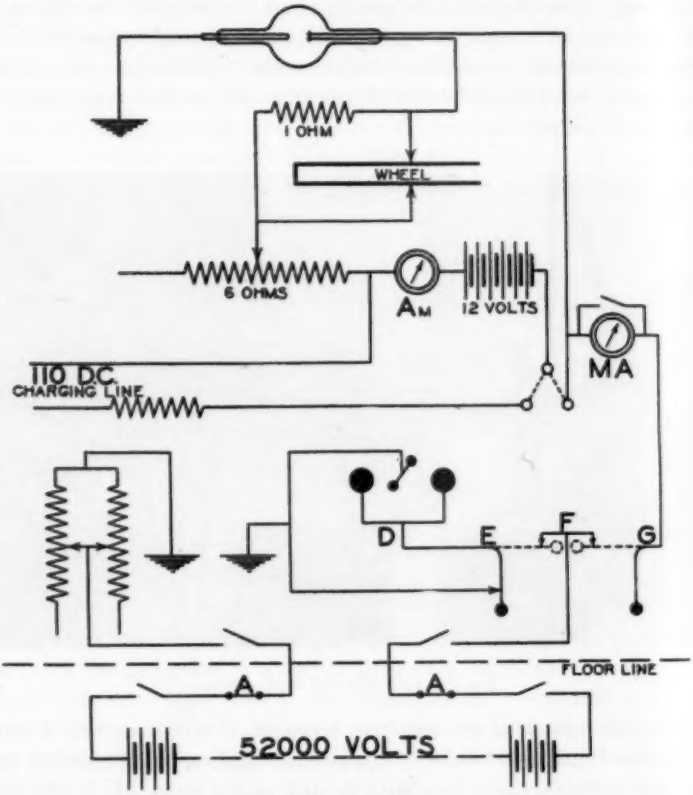

FIG. 6. *Wiring diagram.*

strands of wire. This joint is surrounded by a hollow sphere, several centimeters in diameter, in which a vertical slot is cut. The rod usually hangs vertically down and in that position makes contact with a horizontal rod connected to earth, and discharges the voltmeter. When it is drawn up into a horizontal position it makes contact with the line from the battery through the horizontal rod at *F*, and thus puts the

voltmeter across the battery. (See *E* in Figs. 5 and 6.) The lead from the tube is equipped with a similar switch—shown at *G* in the figures—except that there is no earth connection on the tube switch. These simple switches, operated by enameled silk fish line, have proved very satisfactory.

In all of the high voltage wiring, care has been taken to avoid sharp bends, and at the same time the connections have been made as short as possible. Every free end has been finished with a sphere several centimeters in diameter. The fact that it was possible to use this apparatus during the humid weather of July and August indicates that the insulation is very satisfactory.

Fig. 6 shows diagrammatically the entire wiring scheme except that of the charging system for the large storage battery. The parts lettered *A* to *G* correspond to the similarly lettered apparatus shown in Figs. 3 and 5.

CRUFT HIGH TENSION ELECTRICAL LABORATORY,
HARVARD UNIVERSITY,
JUNE 30, 1925

Polarization of Light scattered by organic Vapors.—The polarization of light scattered by atoms or molecules of a gas irradiated by a plane polarized beam and observed at right angles to the irradiating beam has been studied a number of times. If the molecules contained electrons free to vibrate in any direction, the scattered light thus observed would be plane polarized; imperfect polarization is explained, in terms of the classical theory, by intra-molecular constraints upon the vibrating electron, and in terms of the quantum theory by changes in orientation of the molecule between absorption and re-emission. Ganesan's experiments show that, whatever the cause of the depolarization, it is not one which becomes more potent the more the molecule departs from a simple spherical form; for he investigates series of vapors in which the molecule lengthens out from one to the next, and the depolarization goes down rather than up. There are evident correlations between the amount of depolarization and the nature of the organic compound which should form an interesting field of study. —[A. S. Ganesan, Calcutta; Phil. Mag. *49*, pp. 1216-1222, 1925.]

KARL K. DARROW

Dependence of Metallic Reflection on Temperature.—Nearly all of the very many attempts to detect a variation of the optical constants of metals with temperature have yielded definitely negative results; but Ebeling, concentrating his attention upon the reflecting power of silver for light of the spectrum region covering its well known minimum of

reflecting power (near 3160 A), and using what he describes as an unprecedently high dispersion, finds that the reflecting power at its minimum rises from about 4 per cent to "20 or 30 per cent" as the temperature of the silver rises from 20° to 200° C. At a nearby wave length the reflecting power falls with increasing temperature, so that an examination with weak dispersion might indicate that there was no change at all in this region. At liquid air temperature the reflecting power at the minimum is still lower. Indications of a similar effect were observed with copper near the wave length of its (much less pronounced) minimum of reflecting power. Such variations would be very important if definitely established. The author's attempt to correlate the reflecting power of the solid with the line spectrum of the vapor are not convincing.—[L. Ebeling, Frankfort: ZS. f. Phys. *32*, pp. 489-501; 1925.]

KARL K. DARROW

Continuation and Summary of Aston's Discoveries of Isotopes. —Aston now announces the analysis of ten more elements into their isotopes, bringing to 56 (out of the 80 non-radioactive elements) the total number analyzed. As in the last previous experiments, these most lately conquered elements were used in the form of appropriate halogen salts, incorporated into the anode of a specially designed discharge tube, so that anode rays were formed of their molecules. The newly analyzed elements are In, La, Pd, Bi(each of which has only one sort of atom), Sr (two isotopes), Ba (two), Nd (3 or 4), Ce (2), Zr (3 or 4), Cd (6), Te (3). All the three isotopes of Te are isobaric with isotopes of Xe; one of the isotopes of Cd is isobaric with one of those of Sn, and there is a great similarity between the relative intensities in the groups of isotopes for Cd and for Sn. Perhaps the most curious result from these researches is that some quite large deviations from integer numbers are definitely established for individual isotopes (e.g. 87.8 is the mass of the chief isotope of Sr). Further, when the percentage of the different isotopes of (for instance) Ba is inferred from the relative darkenings they produce upon the photographic plate and the combining weight of Ba is then calculated by taking the weighted mean, it comes out gravely different from the accepted value furnished by the chemists; and there are other such instances. There is a table of all known isotopes. Aston indicates that these may be his last attempts to analyze hitherto untested elements, as he prefers with new apparatus to devote himself especially to determining the masses of the isotopes already discovered. He gives an estimate of the relative difficulty (or ease) which the untested elements may be expected to offer to those desirous of undertaking such work.—[F. W. Aston, Cavendish; Phil. Mag. *49*, pp. 1191-1200; 1925.]

KARL K. DARROW

PHOSPHOR BRONZE HELICAL SPRINGS FROM THE STANDPOINT OF PRECISION INSTRUMENTS*

By W. G. Brombacher

I. SUMMARY

This paper gives the results of an investigation at the Bureau of Standards made for and financed by the Engineering Division of the U.S. Army Air Service in order to study the properties and the design of helical springs. The primary object was to obtain knowledge which would be useful in the design of precision instruments. It was planned to secure a number of sizes of wire of various metals and alloys, to make helical springs of these wires and to study their performance; however, only phosphor-bronze springs have been constructed and tested. It is hoped that the work can be extended in the near future to include steel springs.

In the course of the investigation apparatus was designed and constructed for the purpose of determining the modulus of torsion of the spring wire. The modulus was found for the various sizes of the phosphor-bronze wire. Nine sizes of phosphor bronze springs were designed by aid of the nomograms which had been prepared for the purpose of eliminating the labor of making the computations. See Figs. 1 and 2. Forty-two springs were made, comprising at least three springs of each of the majority of the designs.

The data obtained for the purpose of determining the performance of the springs consisted mainly of measurements of the deflection of the springs under loads which were applied so as to put the springs in compression. The amount and rate of application of the loads were varied in order to obtain desirable data. Data were obtained for a number of springs both before and after modifications which consisted of the successive cutting off of some of their coils.

The performance of the phosphor-bronze springs was classified in the following manner. The experimental and computed stiffnesses were compared; the maximum fiber stresses corresponding to the various loads were computed, with the aim of giving the value of the maximum fiber stress at the proportional limit; the amount of the hysteresis and its distribution with load are given; the amount of the after-effect and

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the drift are considered; the values of the buckling criterion given by Hurlbrink are computed from the data on buckling; and the effects of seasoning are given. An important fact was brought out in comparing the stiffness determined experimentally with that determined theoretically. The springs were from 5 to 21 per cent more flexible than predicted by theory. See Table 6.

A bibliography is given at the end of the paper.

II. INTRODUCTION

Many qualities of springs require consideration in their design for use in a precison instrument. The qualities which are important in certain instruments are unimportant in others, as, for example, the magnetic characteristics of the spring material are vital for electrical instruments while unimportant in others. These qualities for aeronautic instruments such as altimeters, air speed indicators, tachometers and a host of special instruments may be divided into elastic and temperature properties. The elastic properties are designated by (a) stiffness, (b) proportional limit, (c) buckling (for helical springs in compression), (d) hysteresis, (e) drift and (f) after-effect. The temperature properties may be named by the effects to which they contribute in instruments: (g) zero shift, and (h) change in scale value.

(a) Stiffness, defined rigorously as the rate of change of load with deflection, is the one factor which has been given mathematical expression in terms of the constants of the material and spring. Data on the exactness with which these relations hold for various shape springs such as helical, spiral, flat, etc., are largely lacking. The effect of heat treatment of steel springs on the stiffness also requires consideration. Practice in large measure has been to obtain load-deflection data for various shapes of springs and to determine from this a constant which includes the proper elastic modulus of the material and a shape factor, the latter measuring the amount by which the mathematical expression for stiffness differs from the actual stiffness.

(b) The proportional limit of a load-deflection relation of a spring relates to the point at which the deflection ceases to be directly proportional to the load. The value of the maximum fiber stress corresponding to the proportional limit is usually used as a criterion in determining maximum loads.

(c) Buckling of loaded helical springs is here restricted to that curvature of the longitudinal axis which is symmetrical about the central lateral axis. This eliminates from consideration buckling which

is unsymmetrical with the longitudinal axis. A criterion for buckling in terms of the dimensions of the spring has been derived by Hurlbrink.[15]

(d) Hysteresis is defined as the excess of deflection for a given load with loads decreasing over the deflection for the same load with loads increasing.

This quality of a spring is of the most importance in precision instruments. Its value should be a minimum, which is achieved by the proper choice of material and by the design of springs with comparatively low maximum fiber stresses at the maximum loads.

(e) Drift is the increase of deflection with time for a given load.

(f) After-effect is the amount by which the spring fails to return to its original length after a load-deflection cycle.

(g) Temperature zero shift. This is defined as the change in length or change in deflection due to change in dimensions, with change in temperature at zero load. This effect is usually of little importance in instruments owing to the fact that it can be readily compensated.

(h) The change in scale value of an instrument is caused by the change in stiffness of the elastic system with change in temperature. Thus the temperature effect which needs to be chiefly considered is the change in stiffness of a spring with change in temperature.

This paper gives the results of a preliminary investigation in which helical springs made of commercial phosphor-bronze spring wire were tested in compression. It was found that, as was expected, this particular phosphor-bronze resulted in springs which are unsuitable for use in most precision instruments. However, the procedure of testing is outlined and the values of many of the various qualities which may govern the choice of a spring were determined for this material which give the order of magnitude which may be expected for ordinary phosphor-bronze. Further, the correspondence which may be expected between the theoretical and experimental stiffness and the theoretical and experimental buckling criterion is here indicated for the case when no elaborate procedure is followed in the manufacture of the springs.

III. FORMULAS AND CHARTS FOR THE DESIGN OF HELICAL SPRINGS

(a) *Relation between stiffness and number of turns.*

Let

L_D = Load in tension or compression on the helical spring.

D = Deflection of the spring under load L_D.

S = Stiffness of the spring, defined by the ratio $\frac{L_D}{D}$

n = Number of turns or coils of the spring.
a = Radius of the spring wire.
r = Radius of the coils of the spring, taken to the center of the wire.
G = Modulus of elasticity in torsion.
α = Pitch angle of the coils of the spring.
E = Young's modulus of elasticity.

The stiffness of a spring as defined above is restricted in this paper so that it applies only to such deflections as are approximately proportional to the load. Because of this restriction, particular stiffnesses,*[10] such as *instantaneous* stiffness, $\frac{d L_D}{d_D}$, or *initial stiffness*, $\frac{d L_D}{d_D}$, have values approximately equal to the stiffness S here defined. The stiffness S in the sense it is here used is also called the *effective stiffness*.[18]

The pitch angle of the spring, α, is more exactly defined as the angle made by the tangent to the center line of the wire of the spring and a plane perpendicular to the axis of the spring.

If a force or load is applied so that it causes only a compression or or tension of the spring which is symmetrical about the axis of the spring, the relation of the stiffness to the constants of the spring is given by the following equations:[2-4-5]

$$\left.\begin{aligned} &\frac{1}{S} = 2\pi r^3 n\left(\frac{\cos^2\alpha}{A} + \frac{\sin^2\alpha}{B}\right) \\ &\text{where} \qquad A = \frac{\pi G a^4}{2} \\ &\text{and} \qquad B = \frac{\pi E a^4}{4} \end{aligned}\right\} \tag{1}$$

Into the derivation of this formula enter the assumptions that, for a deflection of the spring, changes in both the angle of rise, α, and the radius, r, are small.

It will be noticed upon inspection of the formula that the stresses in the spring are such that the wire is under both torsion and tension. The tension is relatively small for small values of the pitch angle.

Formula (1) is usually modified by assuming that the angle α is small, in which case there results the relation

$$n = \frac{G a^4}{4 r^3 S} \tag{2}$$

* The citations corresponding to the numbers given in this paper may be found in the appended bibliography.

For the design of springs for this investigation formula (2) has been used. The errors introduced by the neglect of the angle α in this customary formula are easily determined. Formula (1) can also be written

$$n = \frac{G a^4}{4 r^3 S} \cdot \frac{1}{(\cos^2\alpha + .8 \sin^2\alpha)}$$

in which it has been assumed that $E = 2.5G$ for spring metals. The value of the term $\frac{1}{\cos^2\alpha + .8 \sin^2\alpha}$ for various values of the angle α are given below:

TABLE 1.

Angles of Rise, Degrees	$\frac{1}{\cos^2\alpha + .8 \sin^2\alpha}$
0	1.000
1	1.000
2	1.000
3	1.001
4	1.001
5	1.002
6	1.002

Thus, it will be seen that the values of the number of turns found from the more exact formula (1) agree with those found from formula (2) within 0.2 per cent for pitch angles up to 6°.

The error arising from the fact that formula (1) includes the assumption that the changes in the radius of the spring coils are proportionately small for a particular deflection, is commonly considered to be avoided when the ratio of the diameter of the spring coils to the diameter of spring wire has a value greater than 4.

(b) *Maximum Fiber Stress.*

For most purposes springs should not be loaded beyond the limit at which the deflection is directly proportional to the load, and for all purposes a limit should be set to the maximum load which can be applied. The maximum load can be determined by a knowledge of the permissible fiber stress in torsion or shear of the spring material used. The maximum fiber stress, f, of the helical spring under load L (references 4 and 5) is given by the equation

$$f = \frac{2Lr}{\pi a^3}. \qquad (3)$$

It is to be noted that this formula is independent of the number of turns of the spring. Thus, a given load gives the same fiber stress whether the spring has ten or twenty turns.

(c) *Buckling.*

Helical springs in compression may buckle. Hurlbrink[15] has developed a formula for determining the buckling loads for helical springs of square and round wire, loaded in compression. This is, for wire of circular cross section

$$B = 45\frac{r^2}{hD} \qquad (4)$$

$$B \geq 6$$

in which h = the height of the spring when not under load.

D = the deflection under a given load.

r = the radius of the coils.

B = the buckling criterion.

The idea is that buckling will not occur if B is numerically equal to or is greater than six.

(d) *Nomograms for Use in Design.*

Two alternate nomograms or alignment charts were constructed, which solve Eq. (2) graphically. The value of G, the torsion modulus was taken as 12×10^6 pounds per square inch in preparing these nomograms. In order to take care of deviations from the chosen value of G, an auxiliary scale is provided which gives a factor corresponding to values of G from 3×10^6 to 14×10^6 pounds per square inch. If the factor corresponding to a particular value of G is multiplied by the number of turns obtained from the nomogram, the product will be the correct number of turns. The alternate nomograms and auxiliary scale are given in Fig. 1.

A feature of this nomogram which is worthy of mention is that no lines need be drawn in order to use it. It is only necessary to obtain the distance between the values on the scales of one line and to measure the same distance on the two scales of the other line. They must be in the proper relative directions, however.

To illustrate the use of this nomogram, assume a spring with the constants: $G=6.5\times10^6$ lbs. per in^2., $n=15$ turns, $S=20$ lbs. per in., $a=.064$ in., r is to be found. From the auxiliary scale of Fig. 1 the factor .54 is found to correspond to the value of G. Since the correct n is already known, the value of n to be used in

the nomogram is found by dividing 15 by 0.54 which gives 28 as the number of turns to be used with the nomogram. Using nomogram

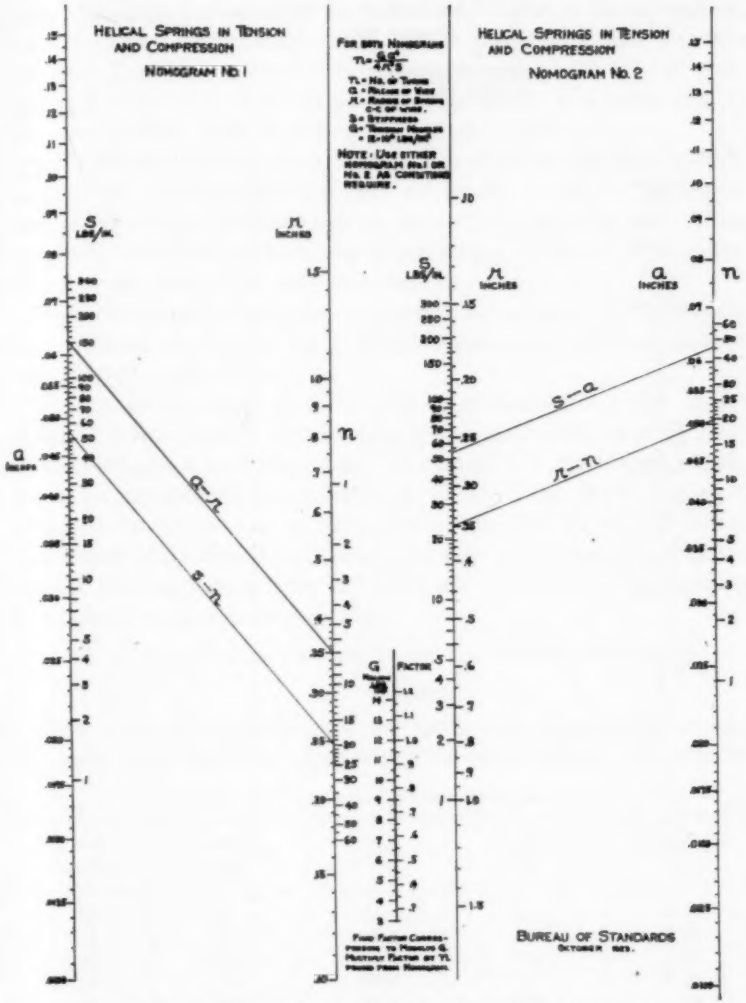

FIG. 1.—*A monogram for use in designing helical springs.*

No. 1 the distance between 20 on the "*S*" scale and .064 on the "*a*" scale is spaced by a pair of dividers. With the same spread, one leg of the dividers is then set on 28 of the "*n*" scale and .46 inches, the

value of r, is read off where the other leg falls. The proper direction from the value of n in which to space r is easily seen by noting that the

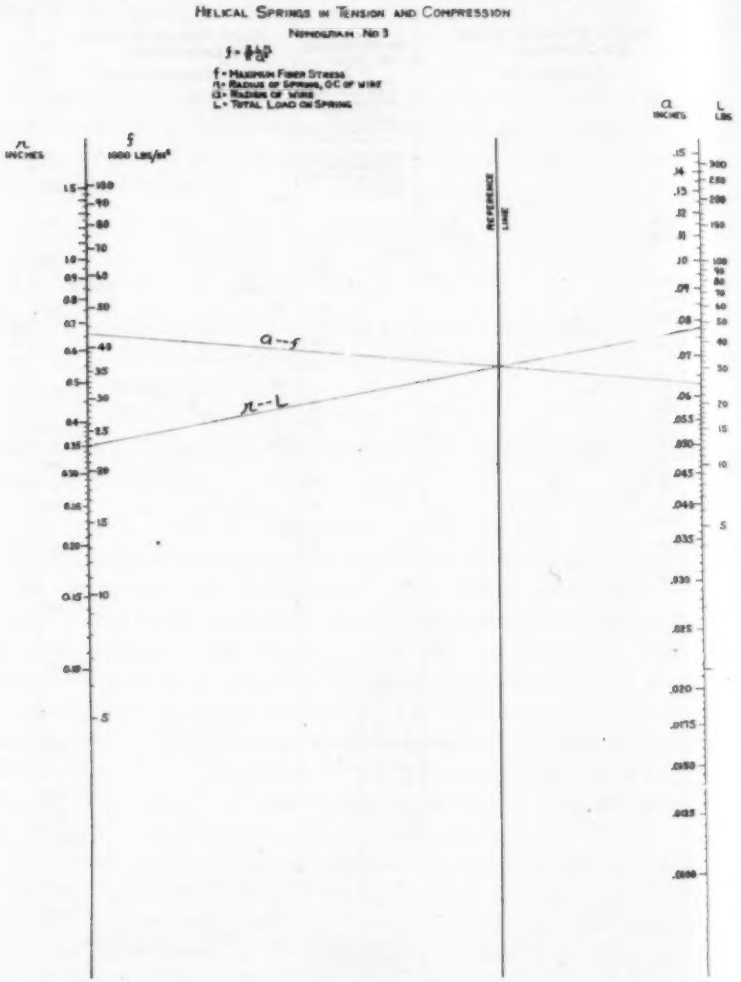

FIG. 2.—*A nomogram which gives the relation between the maximum fiber stress and maximum load.*

a to r and S to n lines must be parallel, or the a to S and the r to n lines must have the same sense of direction.

Nomogram No. 3, Fig. 2 solves graphically Eq. (3) for the maximum fiber stress. In order to use this nomogram lines must be drawn or some equivalent process must be performed. If f is to be determined, connect r and L and through the intersection of the r-L line with the reference line draw a line from a. Its intersection with the "f" scale gives the value of f. The procedure is similar if f is known and L, the corresponding load on the spring is to be found.

The following example illustrates the use of the nomogram. Assume $a = .064$ in., $f = 10{,}000$ lbs. per in^2., $r = .46$ in., $L =$ is to be found. Indicate the point of intersection of the a—f and reference line. Place a ruler so as to line up this intersection with r, then read L as determined by this line. This gives 8.5 lbs.

It may be mentioned that a nomogram could be made for determining the maximum fiber stress which would require only a pair of dividers used similarly as in Fig. 1.

Various writers have given graphs or nomograms for use in the design of helical springs. The references to a number of such are given in the bibliography. References 7 to 10 and 17. In addition to the above, an unpublished chart drawn by K. D. Wood at the Bureau of Standards has come to the notice of the writer. This has the advantage of grouping the stiffness and maximum fiber stress formulas together and the disadvantage of being inflexible as to variations in the modulus of elasticity or maximum fiber stress.

IV. ANALYSES AND CONSTANTS OF THE SPRING MATERIALS

(a) *Analyses*

Table 2 gives the analyses of the spring wire of phosphor bronze. The wire was obtained, without special specifications, through the

TABLE 2. *Analyses of phosphor-bronze wire*

B & S Gage No.	Diameter Inch.	Copper %	Tin %	Phosphorus %
10	.103	95.8	3.8	0.36
12	.0812	95.9	3.6	0.38
14	.0647	95.7	3.8	0.35
16	.0495	95.6	3.9	0.34
18	.0401	95.6	3.9	0.32

Iron: Less than 0.05 per cent in each sample.
Lead, Nickel and Zinc: Not detected.

yearly contracts awarded by the General Supply Committee of the government. It will be noted that it is practically free from zinc, which

is desirable, but contains less than 4 per cent of tin, of which a greater proportion would have been desirable.

(b) *Moduli of Elasticity in Torsion*

The only constant of the material needed to design the springs was the modulus of torsion. There are two customary methods by which this constant can be determined, one consisting in applying a known torque to a sample of the wire and measuring the corresponding twist of the wire. The other method consists in attaching the upper end of a length of the wire from a fixed support, fastening weights to the lower end and then obtaining the period of the system. The latter method was used because it was the more convenient.

The modulus was determined for one sample each of Nos. 12, 14, 16 and 18 phosphor-bronze wire. The results are given in detail in Table No. 3. The average of the determination for each wire was used in the spring design.

For Nos. 8 and 10 phosphor-bronze wire the modulus was estimated to be 6.5×10^6 pounds per square inch. This was done because the amount of wire on hand did not allow determinations to be made.

TABLE 3. *Modulus of torsion for phosphor-bronze wire*

Gauge of Wire	Average Diameter of wire, inches	Modulus of Torsion, G, 10^6 lbs/in^2
12	.0812	6.56
12	.0812	6.59
14	.0647	6.43
14	.0647	6.34
16	.0495	6.44
16	.0495	6.33
16	.0495	6.65
18	.0401	6.71
18	.0401	6.72

Some time after the completion of the main investigation the modulus of torsion was determined for a sample of the No. 8 wire by the alternative method of applying a known torque and measuring the corresponding twist. A mean of two results gave for G the value 6.34×10^6 pounds per square inch which deviates from the assumed value of 6.5×10^6 by 2.7 per cent.

V. DETAILS OF THE CONSTRUCTION OF THE PHOSPHOR-BRONZE SPRINGS

(a) *Method of Construction*

An arbor or mandrel was made somewhat smaller than the required inside diameter of the spring. The wire was wound on the arbor which was held in the head stock of a lathe. The pitch of the spring was determined by feeding the wire to the arbor from the lathe carriage. This was done by hand. The wire was kept under constant tension while being wound by means of weights suspended from the free end which hung over a pulley.

The proper arbor diameter and winding loads were found by a cut-and-try method, until both straight and accurate springs were obtained. For convenience the data on the construction have been combined with other data and are given under sub-head (b) following.

(b) *Data on the Design and the Construction of the Springs*

A variety of springs were designed with suitable variations of the various factors. The nomograms in Figs. 1 and 2 were used for this purpose. At least three springs of each design were made. Nine sets of springs were made. In Table No. 4 design data are given for each set in the order of their construction. The corresponding data on construction are given for each spring in Table No. 5.

TABLE 4. *Design of springs*

Design No.	Wire No.	Diameter of wire inches	Stiffness for 15 turns lbs/inch	Mean Diam. of coils inches	Modulus of torsion 10^6 lbs/inch^2	Load for maximum fiber stress of 10,000 lbs. per inch^2. lbs.
IX	18	.0401	3	.36	6.7	.67
VIII	18	.0401	9	.25	6.7	1.0
VII	16	.0495	20	.25	6.5	1.9
VI	14	.0647	7.6	.50	6.4	2.1
V	12	.0812	18.7	.50	6.6	4.4
IV	10	.103	20	.68	6.5	6.3
III	10	.103	6	1.00	6.5	4.2
II	8	.128	20	.92	6.5	8.5
I	8	.128	7.6	1.25	6.5	6.5

(c) *Analysis of the Data on Construction*

In analyzing the construction data consideration was first given to the winding load. It was desired to obtain a straight spring using a

minimum tension in the wire. The first experiments brought out the fact that a great aid in obtaining a straight spring was to allow the load producing the winding tension to remain on the wire for half an hour after winding the spring. This was done for the remaining springs. Furthermore, a relation between winding load, T, in pounds, and the radius, a, in inches, of the phosphor-bronze wire was determined from the data on satisfactory springs:

$$\log T = 21.2\, a + .53 \tag{5}$$

TABLE 5. *Dimensions of constructed springs*

Spring No.	Mean D.am. of Coils Inches	Winding Load lbs/inch²
IX-9	.364	8300
-10	.363	
-11	.363	
VIII	.25	8300
VII	.25	5400
VI	.50	4600
V	.50	3280
IV	.68	4150
III	1.00	4250
II	.920	4250
I-1	1.472	4250
I-2	1.442	4250

The data indicated that considerable variation from the loads given by equation (5) will still give good springs, however.

It will be noted from the data given for each spring that the stresses in the wire varied from 8300 to 3280 pounds per square inch. These values are well within the proportional limit for phosphor-bronze. It is evident that no simple relation exists between the various winding loads and their corresponding stresses.

An attempt was made to find a relation between the size of the arbor or mandrel and the mean or inside diameter of the coils. The ratio of the arbor diameter to mean coil diameter is approximately constant and equal to about 0.78 for all sizes of wire.

The ratio of the arbor diameter to the inside diameter of the coils was also computed. There was a greater deviation from a mean of the values of this ratio compared with the one previously discussed but no regular relation was indicated when it was plotted against the diameter of the wire. It should be here noted that insufficient data prevent any

conclusion as to the variation of the ratio with the diameter of the coils for springs of the same size wire.

VI. METHOD OF TESTING SPRINGS AND DESCRIPTION OF APPARATUS

(a) *Mounting of Springs.*

Some experimenting to test the possibility of obtaining springs coiled at the ends so as to give a flat end surface which was also perpendicular to the axis of the spring indicated that it would be a long drawn out process to obtain satisfactory results. There was also some uncertainty as to the effect of the turns at the end on the stiffness, but this proved in the light of the experimental results to be of little consequence.

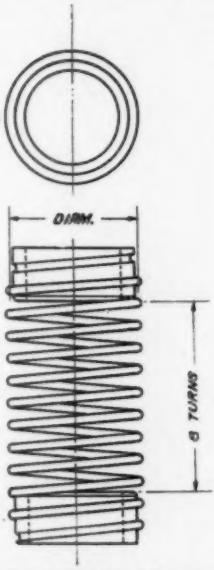

FIG. 3.—*A spring fitted with end plugs ready for mounting in testing apparatus.*

When the springs were made in this way, the flat end coils were of much smaller diameter than were the other coils, and were not counted in determining the number of turns. Finally, all of the springs were tested with brass plugs in both ends, threaded to the pitch of the springs. The ends were finished square with the axis of the spring. See Fig. 3. It can be readily seen that the length of the wire taking part in the deflection is thus made perfectly definite, thereby making it easy to count the number of coils taking part in a deflection.

(b) *Description of the Apparatus.*

The springs were placed upon a metal plate which was supported by a substantial wooden framework. A flat plate and rod rested on top of the spring, as clearly shown in Fig. 4. Another rod was suspended from the above-mentioned rod. Weights were hung from the stirrups of the lower rod. These consisted of small manilla envelopes filled with lead shot weighed accurately to constitute 1/8, 1/4, 1/2, and 1 pound loads. A hole punched in the sealed envelope was given a metal rim by

SPRING TESTING APPARATUS.

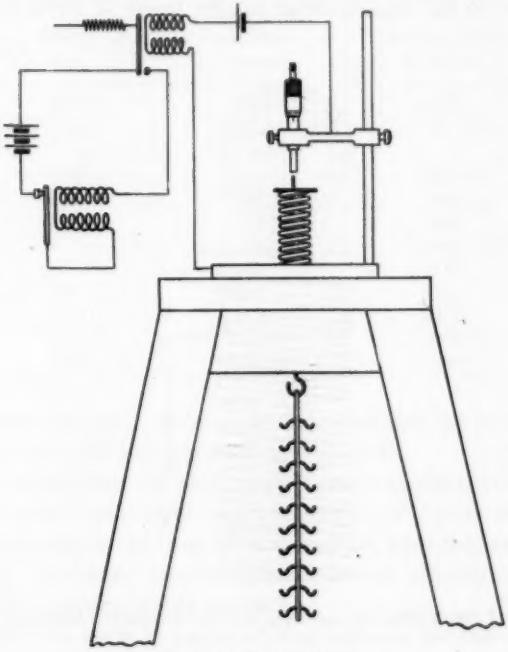

FIG. 4.—*Testing apparatus.*

pressing into it a patented paper fastener. The weights of the envelopes were adjusted to allow for this procedure. It was found best always to place the loads on the stirrups symmetrically.

The deflections were measured by a micrometer head, connected by a steel rod to the base upon which the spring rested. The contact between the top bearing rod and the micrometer was determined by a relay and buzzer system, as shown schematically in Fig. 4. This

scheme was adopted in order to reduce the current between the contacts to a minimum. Silver contacts were used, obtained by flowing silver on both the bearing plate rod and the micrometer tip. The micrometer top was finished off so as to give a flat surface, while the bearing plate rod had a hemispherical tip. This obviated the usual difficulties due to oxidation of the contacts since silver oxides are good conductors of electricity. This arrangement was found entirely satisfactory for all tests save those for determining hysteresis.

(c) *Outline of Tests.*

The springs were tested only in compression. A typical example of the procedure is the measurement of the deflection of the spring for one half pound increments of load up to five pounds maximum load and then back to the initial load by decrements of the same magnitude. The initial load was in all cases very nearly one eighth pound. Readings were estimated to the nearest .001 millimeter but were required to this order of accuracy only for obtaining the drift and hysteresis. The initial tests were made in order to test the method. A discussion of the results of these is given directly below. Those tests which determined the properties of the springs are considered later.

The effect of the rate of change of the load was considered. It was found easy to run tests so that readings could be taken at time intervals of between one and two minutes. The change in stiffness for any variation between these time intervals between readings was of the same order of magnitude as the experimental error. In fact this was found to be true for a test in which the readings were taken at one minute and five minute intervals. It is therefore believed that this factor does not enter into any comparisons made between tests.

The influence of the manner of applying the load and of the magnitude of the increments was also considered. It is currently believed that the best manner of applying the load is by very small increments such as may be obtained by allowing small lead shot to flow on the load pan through a funnel. This method was not found practical for this investigation, however. It was decided to vary the load by increments of 1/8, 1/4, 1/2 or 1 pound. There was no sensible difference in the deflections of a spring for a given load for increments of either 1/8 or 1/4 pound or in other cases of either 1/2 or 1 pound. The envelopes containing the shot were attached to the stirrups by hand. It was noted that it was necessary to place the envelopes with care so that the spring took up the load without impact. Considerable trouble was experienced in

taking readings because of the vibration of the springs. The insertion of the lower end of the stirrup rod in a beaker of heavy oil practically eliminated the trouble.

VII. PERFORMANCE OF SPRINGS

(a) *Performance factors*

For all of the purposes for which springs may be required it is necessary to know the limits of the elastic errors of a particular design of spring for a given stiffness and also the magnitude of the load which the spring will carry safely. In this investigation certain springs were designed and the predicted stiffnesses were compared with those of the actually constructed springs. The sources of the discrepancies between the theoretically-predicted and the experimentally-determined stiffnesses will be discussed in the following section. The load which an instrument spring should carry is determined by the maximum fiber stresses, and these have their limit determined by the limit of the proportional deflection of the spring.

There are certain other performance factors which are not so easily determined theoretically but which are of great importance in the use of springs in measuring instruments. These are (1) hysteresis, (2) after-effect, (3) the drift, all of which are defined in the Introduction. For helical springs in compression, (4) buckling is an important factor. The effect of (5) seasoning on the performance of the springs is also considered.

(b) *Stiffness*

As has been previously stated, helical springs in tension and compression are designed from the formula

$$S = \frac{G a^4}{4nr^3} \tag{2}$$

The quantity G is a constant of the material, the quantity a is controlled in manufacture of the material, while n and r are controlled during the manufacture of the spring. Thus the stiffness, S, is computed by equation (2) and is compared with the values found experimentally from the load-deflection curves. A typical load-deflection curve, selected from the one hundred and forty tests, is given in Fig. 5. In addition to the experimental data the figure contains for comparison the theoretical load-deflection curve derived from equation (2). The stiffnesses of the springs determined from the load-deflection data,

the computed stiffness in each case, and the average ratio for each spring group of the stiffness to the computed stiffnesses are given in Table 6.

The fact noticed by an examination of Fig. 5 is that the computed stiffness is greater in all cases than the experimentally determined value. It may be seen by an examination of Fig. 6 and

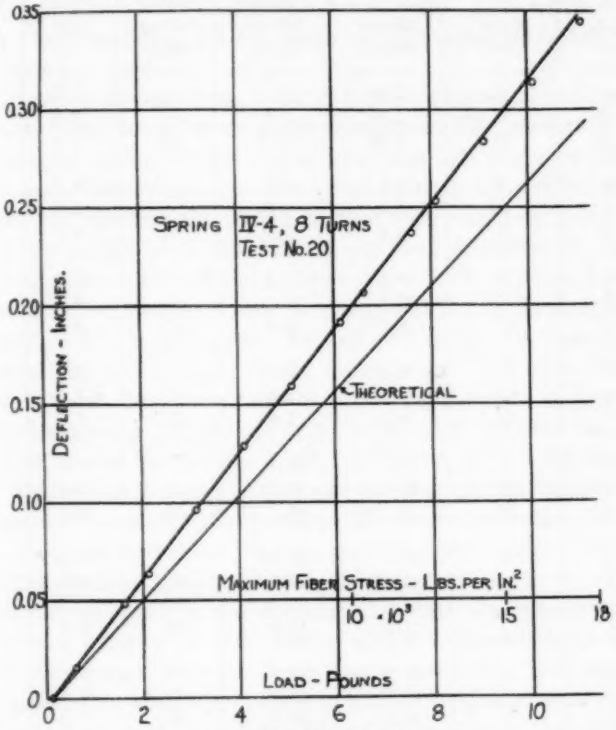

FIG. 5.—*A typical load-deflection curve.*

Table 6 that this holds for all of the springs constructed and tested. In connection with the discussion of this fact the following possible causes will be considered: (1) Assumptions underlying formula (2), (2) End effects, (3) Inaccuracy of value for the modulus of torsion, (4) Other causes.

(1) The assumptions underlying formula (2) are that changes in the radius of the spring and in the pitch angle are small and that the pitch angle of the spring is small. Table 1 given with the discussion on the

relation between stiffness and number of turns, shows that for pitch angles as high as 6° the increase in stiffness is only 0.2 per cent. Any effect of large changes in the pitch angle during loading would be manifested by deviations from a straight line of the load-deflection

TABLE 6. *Performance of Similar Springs of Phosphor Bronze*

Ident. No.	No. of Turns	Stiffness lbs/in.	Computed Stiffness lbs/in.	Average Stiffness Ratio	r/a	Pitch Angle degrees
IX-9	15	2.54	3.0		9.0	4.5
-10	15	2.70	3.0		9.0	4.5
-11	15	2.66	3.0	.88	9.0	4.5
VIII-1	15	8.5	9.0		6.2	5.0
-2	15	8.9	9.0		6.2	5.0
-3	15	8.4	9.0	.95	6.2	5.0
VII-1	15	18.4	20.0		5.0	5.0
-2	15	18.5	20.0		5.0	5.0
-3	15	18.5	20.0	.92	5.0	5.0
VI-5	15	6.2	7.6		7.7	5.0
-6	15	6.3	7.6		7.7	5.0
-7	15	6.2	7.6	.82	7.7	5.0
V-2	15	17.7	18.7		6.1	5.0
-3	15	16.4	18.7		6.1	5.0
-4	15	18.2	18.7	.95	6.1	5.0
IV-2	8	30.7	37.5		6.6	5.5
-4	8	31.6	37.5	.83	6.6	5.5
III-4	7	11.3	12.9	.88	9.7	4.5
II-1**	11	16.8	20.4	.82		
-2	5	47.2	60.0	.79	7.2	4.0
-3	8	30.0	37.5	.80	7.2	4.0
-3	5	*48.0	60.0		7.2	4.0
I-1	8.25	7.32	9.00	.81	11.5	2.4
-2	8	8.25	9.82	.84	11.2	2.5

* Extrapolated from value for 8 turns.
** Without end plugs.

curves obtained for the springs. The curves, however, are straight lines with only one exception which is given in Fig. 5 in which is shown that the spring became slightly stiffer for loads above 8 pounds. This may have been caused by the change in the pitch angle with increase

in load. In this case the pitch angle varied from 5.5° for no load to 4.6° for the maximum load of 11 pounds. It is concluded from the experimental evidence that the difference between computed and experimental stiffness cannot be explained by taking account of the pitch angle.

There is no marked variation from a straight line in the load-deflection data obtained from a large number of tests. This seems decisive evidence that the loads did not cause excessive changes in the radius of the spring.

(2) In order to determine if an end effect existed, several springs were each tested for different numbers of turns. The number of turns was varied by cutting off a portion of the spring, thus gradually reducing the number. The stiffness was determined for each value of the number of turns. In plotting the results so that they might be most easily studied, the reciprocal of the stiffness was plotted against the number of turns in order to give a straight line relation. The computed and the experimentally-determined results are shown in Fig. 6. If an end effect does exist and if it is constant for any one spring, its magnitude is determined by the ordinate of the curve when the reciprocal of the stiffness is zero. In three cases out of the eight shown in Fig. 6 this ordinate has a value of about one turn; in the other five cases the experimental curve goes through the origin. If the end effect is the sole cause of the discrepancy between theory and experiment, the computed and experimental curves should be parallel. This is the fact in the three cases which exhibit an end effect, while in the five cases in which the curves pass through the origin, the slopes are divergent.

Since this evidence is not consistent, it does not appear probable that there is actually an end effect of sufficient magnitude to account for the discrepancy between theory and experiment. This conclusion seems reasonable when the manner in which the springs were mounted is taken into account. The threaded end plugs prevent the ends of the spring from contributing to the deflection. In fact, however, the end effect is not large for the case where, instead of using the end plugs, the final turn of the wire was made of smaller radius and flat. Spring No. II-1 is a spring of this type without end plugs, the ratio of the experimental to computed stiffness of which is given in Table 6. Comparing its value with that for Springs No. II-2 and II-3, no substantial difference is noted.

(3) The modulus of torsion used as a constant of design was determined for a sample of the wire of each size, except Nos. 10 and 8. The

value for wire No. 8 was determined by the static method, as previously given in detail. The values found vary about 4 per cent from wire to wire, which is fair agreement, when the possible variations in manufacture are considered. It is possible that the process of winding the springs may have affected the value of the modulus of the wire in the

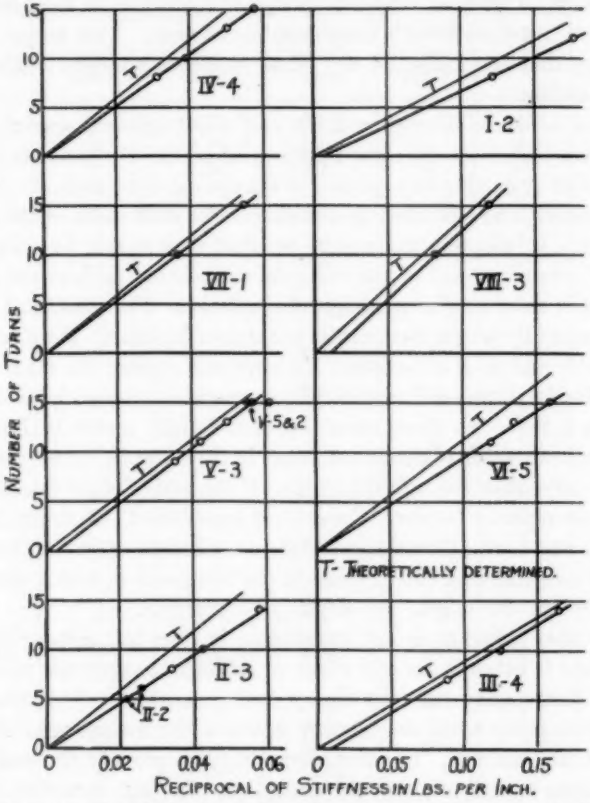

FIG. 6.—*Contrast of theoretical and experimental relation between reciprocal of stiffness and number of coils of spring.*

spring. There is no experimental evidence available bearing on this point, but it would be surprising if it is found that the effect is as great as the difference between the theoretical and experimental stiffness.

(4) *Other Causes.* In the search for other causes an attempt was made to find a possible relationship between the ratio of the experimental stiffness to the computed stiffness and the dimensions of the

spring. To this end the values of this ratio, of a, the pitch angle, and of r/a the ratio of the radius of the coils to the radius of the wire are included in Table 6. It will be readily noticed that no relation exists between the stiffness ratio and the diameter of the wire or the pitch angle. Spring groups III and IV, also I and II have the same diameter of wire. The graph of the stiffness ratio against the radius of the spring for these four groups seems to have promise of results, but plotting these variables for other spring groups gives chaotic results. There is further to be considered a possible relation between r/a, the ratio of the spring diameter to the wire diameter, and the stiffness ratio. It is currently stated that the diameter ratio should not be less than four in order to keep the effect of the neglect of the shearing forces[5] in the design formula down to moderate limits, say 2 per cent. However, group I has a diameter ratio of 11.2, yet the stiffness ratio is only .84, while group VII has corresponding values of 5.0 and .92. The ratio does not seem to be a good criterion.

It was believed that the discrepancy might be due to the effect of internal stresses set up during the winding of the springs. In order to relieve the internal stresses, two springs were heat treated. The stiffness of the springs was determined just before heat treatment. The procedure consisted in heating the springs in an oil bath for one half hour at 210°C and for twenty-one hours at 185°C. The stiffnesses were then determined. The springs were then kept at 220°C for seven hours. The stiffnesses were again determined. The springs seemed "dead" after the heat treatment.

The effect of the heat treatments on the stiffness is presented in Table 7. No changes in stiffness were found of value sufficient to account for the exceptionally large gap between theory and experi-

TABLE 7. *Effect of heat treatment on stiffness*

Spring Ident. No.	Date of Test	Heat Treatment	Increase in Stiffness %
II-2	Feb.-24	none	0
II-2	12-27-24	none	−3.4
II-2	1-10-25	210°C	+2.3
II-2	1-24-25	220°C	+2.3
IV-4	Feb.-24	none	0
IV-4	12-27-24	none	−1.9
IV-4	1-10-25	210°C	0
IV-4	1-24-25	220°C	+2.8

ment which exists in the case of these two springs. It is believed that the temperature of the springs during heat treatment was sufficiently high to release the internal strains.

Whatever the cause, the springs are more flexible than predicted by the design. In the light of the data of Table 6, the formula (2) would modify to

$$S=\frac{G\,a^4}{4.5\,n\,r^3} \tag{6}$$

which in view of the previous discussion may apply to springs of other materials of the same general design and size.

(c) *Limit of Proportional Elasticity. Maximum Fiber Stress.*

It will be recalled that the maximum fiber stress is determined by Eq. (3) but that no limiting value has yet been given f. It was hoped to set the value experimentally, fixing the load L at the limit of proportional elasticity. This value was chosen for the maximum fiber stress because it is of the greatest importance in the design of springs for use in precision measuring instruments.

The maximum fiber stresses corresponding to the loads are given along the x-axis of the load-deflection curve of Fig. 8. The maximum load corresponds to 17,000 pounds per square inch maximum fiber stress. The fact that all load-deflection graphs obtained were straight lines indicates that the proportional limit was not reached in the tests. The maximum load corresponded to maximum fiber stresses of range 15,000 to 18,000 pounds per square inch. This is due to the fact that the available springs buckled with any increase in the maximum loads applied. Lack of time prevented cutting down the number of turns of the springs in order to obviate buckling, or the making of tests with the springs in tension. Handbooks give 20,000 pounds per square inch as the maximum fiber stress at the proportional limit for phosphor-bronze wire in torsion.

(d) *Hysteresis*

Since the method of measurement needs some modification before satisfactory results can be obtained on hysteresis, the data given in Fig. 7 and 8 are not considered final. The ordinates of the curves or the hysteresis are the differences between the deflections for increasing and decreasing loads. Curves 1 and 2, 3 and 4, and 7 and 8 differ only in that in the first of each pair all the experimental values are shown and in the second the experimental values for each load are averaged. The data given are from tests on two similar springs.

The data show that the maximum hysteresis is not independent of the maximum load. Table 8 indicates this more clearly. The data for this table are obtained from Fig. 7 and 8.

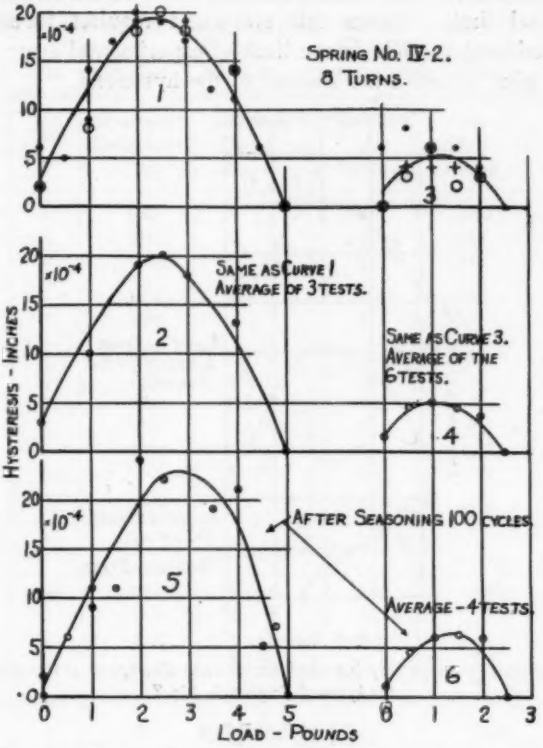

FIG. 7.—*Hysteresis curves for a phosphor bronze helical spring for two maximum loads before and after seasoning.*

TABLE 8

Maximum Load Pounds	Maximum Hysteresis Inches H_m	Maximum Deflection Inches D_m	$\frac{H_m}{D_m}$
2.5	.0005	.08	.006
3.5	.0010	.11	.009
5.0	.0020	.16	.012

H_m/D_m is the ratio of the maximum hysteresis to maximum deflection. Here its values plotted against maximum loads give a straight line which passes through the origin. This is a wholly unexpected result, as the ratio is commonly believed to be a constant for loads within the proportional limit. Hence this statement requires further experimental verification. Within the limit of experimental error, seasoning does not seem to affect the amount of the hysteresis.

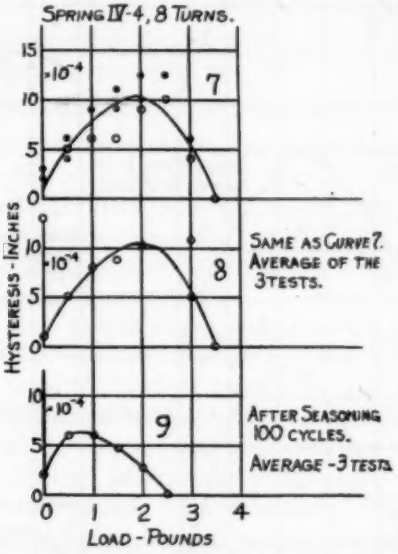

FIG. 8.—*Hysteresis curves for a spring which has the same dimensions as the spring for which the hysteresis is given in Fig. 7.*

(e) *After-Effect*

The after-effect is defined as the algebraic difference in the readings for zero load at the start and conclusion of a load cycle, the final zero reading being the positive one. The data are given in Table 9. These indicate conclusively that the after-effect decreases markedly with successive load cycles. The cycles followed one another closely and were so run that deflection readings were taken at intervals not exceeding two minutes. Thus, for spring I-2 the after-effect decreased from .0010 to .0007 inch in the second cycle and to .0001 inch for the third cycle.

A comparison of the data for spring No. IV-2 before being seasoned and after being seasoned (for definition see section on miscellaneous performance factors) indicates similar reduction in after-effect.

Since the data show that there is an after-effect for the first load-deflection cycle for a given maximum load and that this after-effect diminishes or disappears for successive load-deflection cycles with the same maximum loading, the question arises whether the after-effect is due entirely to the manner of testing the spring or whether it may be

TABLE 9. *Data on after-effect. Time of tests as short as possible. Springs not seasoned.*

Spring Number	Test No.	Maximum Deflection Inches	Maximum Load Pounds	Number of Turns	After Effect Inches	Time of Start
I-2	1	.4906	4	8	.0010	3:20 PM
	2	.6117	5	8	.0007	4:12 PM
	3	.4891	4	8	.0001	4:33 PM
I-2	1	.1522	6	6	.0011	3:11 PM
	2	.1508	6	6	.0001	4:02 PM
IV-2	1	.0813	2 1/2	8	.0006	1:41 PM
	2	.0810	2 1/2	8	0	2:02 PM
	3	.0810	2 1/2	8	0	2:10 PM
	4	.0809	2 1/2	8	0	2:19 PM
	5	.0811	2 1/2	8	−.0002	2:27 PM
	6	.0807	2 1/2	8	0	3:30 PM
IV-4		.0792	2 1/2	8	.0002	
			Springs seasoned			
IV-2	1	.0828	2 1/2	8	.0007	2:32 PM
Seasoned	2	.0821	2 1/2	8	0	2:41 PM
200 times	3	.0822	2 1/2	8	0	2:54 PM
	4	.0824	2 1/2	8	0	3:02 PM
Note: Compare the data on spring IV-2 for the seasoned and unseasoned condition.						
IV-4	1	.0797	2 1/2	8	.0005	9:34 AM
Seasoned	2	.0795	2 1/2	8	0	9:43 AM
100 times	3	.0793	2 1/2	8	.0002	9:53 AM

partly an elastic property of the spring. If it is entirely caused by the manner of making the test, the amount of the after-effect should be of the same order of magnitude for springs of all materials. Comparing the results for phosphor-bronze springs given in Table 9 and for a non-magnetic steel spring tested while this investigation was in progress, it was shown that the after-effect for the non-magnetic steel spring is decisively greater than for springs of phosphor-bronze. It was concluded on this evidence that the after-effect is also an elastic property of the spring. Furthermore, a consideration of the change in after-effect

of the same spring when the maximum load is successively increased strengthens this conclusion. The after-effect, if due to the manner of testing, should be substantially zero when further load-deflection cycles are run with increased maximum loading, because of the lack of play in the parts of this testing apparatus which might contribute to the deflection. For phosphor-bronze springs, increases in the maximum load give rise to an after-effect the amount of which is of the same order of magnitude as the experimental error, but other evidence on this point, the result of tests of a nonmagnetic steel spring, shows conclusively that the after-effect is also an elastic property of the spring, as well as being due to the manner of testing.

In the case where the maximum load is held on the spring for any length of time, causing an increase in the deflection with time, which is defined as drift, an increase in the usual amount of the after-effect would be expected. While no data are available from this investigation, this increase is a well established fact. Its significance is that the rate of recovery from drift is of the same order of magnitude as the rate of drift.

(f) *Drift*

This quantity is determined with somewhat more certainty than hysteresis or after-effect since the experimental conditions remain constant. The change in the deflection with time under a given load is the drift. The data obtained are given in Table 10 and indicate that the ratio of drift for a given time to the total deflection caused by the

TABLE 10. *Drift data*

Spring Number	Load Pounds	Deflection Inches	Drift Inches	Time Hours	Ratio, Drift to Deflection	Number of Turns
I-2	4	.489	.0019	17	.0039	8
IV-4	8	.2529	.00045	1	.0018	8
	5	.1604	.00028	1	.0017	8

load is a constant and approximately amounts to 0.2 per cent for a drift of one hour for the springs of phosphor bronze. The data also indicate that the drift ratio increases with time. If the law that the drift ratio varies as the cube root of the time as was first suggested by M. D. Hersey be used to find the drift ratio for one hour from the value for 17 hours which is given for spring No. I-2, the value .0015 is obtained. The amount of data is small but is believed to be trustworthy owing to relatively great freedom from experimental error.

The drifts for spring No. IV-4 were obtained from a curve of the drift-time observations.

(g) *Buckling*

A considerable amount of data on buckling was obtained, as all of the springs buckled even when loaded below the proportional limit. Some of the buckling was caused by the faulty centering of the top plate and connecting rod or by the end plugs failing to be perpendicular to the axis of the spring. These causes were especially evident for long springs of small diameter. A consideration of the problem strictly from the theoretical standpoint would necessitate the ruling out of data in which the above causes seemed to operate, but practically these causes cannot be eliminated. Table 11 gives the data on buckling. Most of the buckling loads are for the case where the spring buckled decisively, but in some cases it is believed that a shift of the position of the top plate would have eliminated the buckling. This would not be true buckling. The effect of this accidental cause can be estimated by noting the variation in the buckling loads for groups of similar springs of the same number of turns.

Values of B have been computed for the available data, using Hurlbrink's formula, Eq. (4). These are given in Table 11. It is peculiar that the value for B is somewhere near 6, the criterion number given by Hurlbrink, for the springs of small radius, groups V to IX, for

TABLE 11. *Data on buckling*

Spring Number	Number of Turns	Criterion B	Spring Number	Number of Turns	Criterion B.
IX-10	15	3.8	V-3	9	8
-11	15	4.7	-4	15	7
VIII-1	15	6	IV-2	8	7
-2	15	9	-2	8	8
-3	15	14	-4	15	15
-3	10	10	-4	13	19
VII-	15	10	-4	10	19
-2	15	8	-4	8	9
-3	15	6.5	-4	8	8
VI-5	15	9	III-4	14	15
-5	13	8	-4	10	26
-6	15	9	-4	7	13
-7	15	5.5	II-3	14	28
V-3	13	9	-3	10	32
-3	11	9	I-2	8	16

which the data were thought to be somewhat uncertain, while for the remaining groups it deviates widely from the value 6. The values for springs IV-2 and IV-4 with 8 turns are the most reliable, since these springs were selected for repeated tests and here the values of B are both consistent and average well with those found for spring groups IX to V.

It is suggested that the criterion number be changed from 6 to 10. This new criterion number would apply only to helical springs of round wire and would be restricted to springs which are supported and loaded in the same manner as here described.

(h) *Miscellaneous Performance Factors*

It is well known that the stiffness of springs is affected by temperature changes. In these tests the temperature was considered constant. The range of the room temperature from test to test was from 21° to 27°C, and usually varied less than 2 degrees during any one test.

The seasoning of some of the springs has been mentioned. This consisted of deflecting the springs by hand for the stated number of cycles. The deflections corresponded to a maximum fiber stress of approximately 15,000 pounds per square inch in two cases and 20,000 pounds per square inch in the other case. Some of the effects of seasoning have already been considered. One further effect may be mentioned, which is the effect on the stiffness, data for which is given in Table 12.

TABLE 12. *Effect of seasoning on stiffness*

Spring Number	Stiffness Unseasoned lbs/in.	Stiffness Seasoned lbs./in.	Number of Cycles
I-2	.2	8.15	210
IV-2	30.7	30.0	200
IV-4	31.6	31.4	100
IV-4	31.6	31.4	800

The above table shows a slight decrease in stiffness with seasoning.

An unusual load-deflection cycle was obtained after keeping spring I-2 loaded with 4 pounds for 17 hours. The cycle started with the loads on the spring decreasing and ended with them increasing. The stiffness, hysteresis and after effect were the same as for the load cycle in the usual direction, as would be expected.

It is desired to acknowledge the assistance of Mr. Don F. Sutton in obtaining data during the construction of the springs, of Prof. T. B.

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BUREAU OF STANDARDS,
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MAY, 1925

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Miscellaneous Articles

[16] M. D. Hersey, *Diaphragms for Aeronautic Instruments*. National Advisory Committee for Aeronautics, Report No. 165, 1923.

Influence of Temperature on Intensity of Reflection of X-rays.—On account of the thermal agitation of the atoms of a crystal, it is to be expected that reflection of x-rays at any of the critical glancing angles will become steadily less efficient as the temperature is raised. It is an extremely difficult matter to theorize about, as assumptions must be made regarding the statistical distribution of amplitude and phase among the vibrating atoms of each atom plane, and intricate deductions from the assumptions. Debye's theory terminates in the result that the intensity of reflection, for a beam of wave length λ reflected at its critical glancing angle θ, should vary as $\exp(-a\,T\sin^2\theta/\lambda^2)$ so long as the temperature T remains well above the critical temperature specified by his theory of specific heats for the crystal. James' experiments on rock salt involved two values of λ and several of θ (obtained by using different orders of reflection) and temperatures from 15° to 650° C. The data must unfortunately be subjected to a long series of corrections for other effects of temperature than the one desired, as for instance the thermal expansion of the crystal. The eventual result is that the reflection varies as $\exp(-a\,T^2\sin^2\theta/\lambda^2)$; that is, the dependence on wave length and incidence is correctly given by the theory, but the dependence on temperature is not.—[R. W. James, Manchester; Phil. Mag. *49*, pp. 585-602; 1925.]

KARL K. DARROW

More Work on the "Osglim" Lamp.—This paper refers chiefly to the fact that when a condenser is charged to a sufficiently high voltage and then connected across the terminals of an "Osglim" lamp, a glow discharge commences and continues until the charge on the condenser sinks to a certain critical value. This critical value is studied as a function of the capacity of the condenser (its initial voltage being left unchanged) with which it rises, at first swiftly and then more slowly; and of the initial voltage of the condenser (its capacity being left constant) with which it rises if the capacity is large, but diminishes with rising voltage if the capacity is small. There are also curves representing the minimum value of initial voltage required to start the glow, as functions of the capacity; they resemble equilateral hyperbolae to some extent, the voltage diminishing as the capacity increases. The author's idea is that the glow does not start until a certain fixed amount of the charge on the condenser has been dissipated in a dark discharge through the gas, and that after it ceases another dark discharge continues for a while. The mathematical theory developed by the authors is more elaborate than the observations warrant. It is not clear whether such observations upon a special type of tube (the "Osglim" tube is a peculiarly arranged one, with a filling of 75% Ne and 25% He, although in some of these experiments air was used as the filling) can be regarded as having a general significance.—[J. Taylor and W. Stephenson, Newcastle-on-Tyne; Phil. Mag. *49*, pp. 1081-1103; 1925.]

KARL K. DARROW

SILICA TO GLASS AND TO METAL JOINTS

By L. J. Buttolph

The art of joining common glasses to metal has been highly developed as outlined by Kraus,[1] McKelvy and Taylor,[2] Dundon,[3] Meyers,[4] Ridyard,[5] and Houskeeper.[6] The purpose of this paper is to similarly outline the practical methods of joining silica to common glass and to metal.

There is not available a metal having nearly enough the same coefficient of expansion, over a practical temperature range as silica to permit the direct sealing together of the two materials. If such a metal were available it would also need to be one withstanding the working temperature of silica, about 1700° C., preferably without much oxidation. Table 1 gives the average coefficients of expansion, a, and the melting or softening temperatures, C, of some of the more common materials entering into the construction of silica-to-glass-to-metal joints.

Table 1.

	a	C.
Fused silica	$.518\times10^{-6}$	1700
Corning G702EJ	3.2	800
Corning G702P	3.6	700
Jena Glass (59III)	5.2	—
Soda Lime Glass	8.5	550
Lead Glass	8.9	500
Nickel 36%—steel 64%	.93	—
Tungsten	4.4	3350
Molybdenum	5.15	2550
Platinum	9.55	1755
Steel	10.5	1530
Nickel	13.16	1452
Copper	18.15	1083
Lead	29.5	327

The earliest form of quartz-to-metal joint seems to have been a ground joint to a tapered nickel-steel (Invar) pin patented by Stein-

[1] Kraus, U. S. Pat. 1,046,084; 1912.

[2] McKelvy & Taylor, "Glass-to-Metal Joints," J. Am. Chem. Soc. *42*, p. 1364; 1920.

[3] Dundon, "A Glass-to-Metal Joint," J. Am. Chem. Soc. *45*, p. 716; 1923.

[4] Meyers, "A Glass-to-Metal Joint," J. Am. Chem. Soc. *45*, p. 2135; 1923.

[5] Ridyard, "A Metal-to-Glass Joint," J. Am Chem. Soc. *46*, p. 287; 1924.

[6] Houskeeper, "The Art of Sealing Base Metals through Glass," J. Amer. Inst. of Elec. Eng. *42*, p. 954; 1923.

metz[7] and also described by Küch and Retschinsky.[8] Over a temperature range from 0 to 200 C. nickel-steel has a coefficient of expansion sufficiently constant and near to that of quartz to form a satisfactory ground-joint of the type shown in Fig. 1. At higher temperatures the Invar coefficient increases and the greater expansion relative to the silica is liable to shatter the latter. It is obvious that the quality of this joint is determined largely by the care in grinding.

For vacuum work the joint must be sealed on the outside preferably with mercury and on both sides of the joint if the internal pressure is to exceed the external. The simple sealing is illustrated in Fig. 1A with the mercury in turn sealed in place with a cement, while Fig. 1B shows the later method of double sealing. Fig. 1C shows the latest form of the double sealed Invar inlead joint with gas trap. These joints have been successfully used in commercial apparatus and the one shown as Fig. 1C is a type of seal used in the United States and Germany in some of the quartz mercury arcs manufactured today.

Where the Invar pins are available this is a satisfactory joint for many laboratory purposes, as the grinding may be done with the aid of a drill press or lathe, using any of the standard abrasives, down to a fineness of ground silica surface just short of transparency. Aside from its adaptability to current in-leads of high capacity such a joint may itself be disassembled for cleaning or change in apparatus design.

A similar mercury sealed joint, Fig. 1D, was developed by Bastian[9] using a platinum or platinum-iridium (90-10) wire fitted as tightly as possible to a silica capillary with flanged out ends into which the projecting ends of the tungsten wire were melted down to form a tapered plug drawn tightly into the conical ends of the capillary by the subsequent contracting of the wire. This type of inlead joint is obviously even more limited than the Invar pintype to those cases where mercury may be used as a secondary means of making electrical connections as well as a means of making the joint vacuum tight. The necessity of placing the wire under tension resulted in short life and the joint had little practical use.

Kent and Lacell[10] varied the Bastian type of joint by substituting molybdenum and tungsten for platinum, and fusing the capillary into contact with the wire in a vacuum to prevent oxidation, Fig. 1E. The

[7] Steinmetz, U. S. Pats. 910, 736 and 910, 969. (Appd. for 1902 and issued 1909).
[8] Küch and Retschinsky, Ann. d. Phys. *20*, p. 563; 1906.
[9] Bastian, Brit. Pat. 21,383; 1905 and U. S. Pat. 982,119; 1911.
[10] Kent and Lacell, Brit. Pat. 24, 482; 1911.

variation lay in the possibility of using a wire of one half the cross section as compared with platinum for a given current capacity, and the much smaller coefficient of expansion. It is necessary to seal this joint

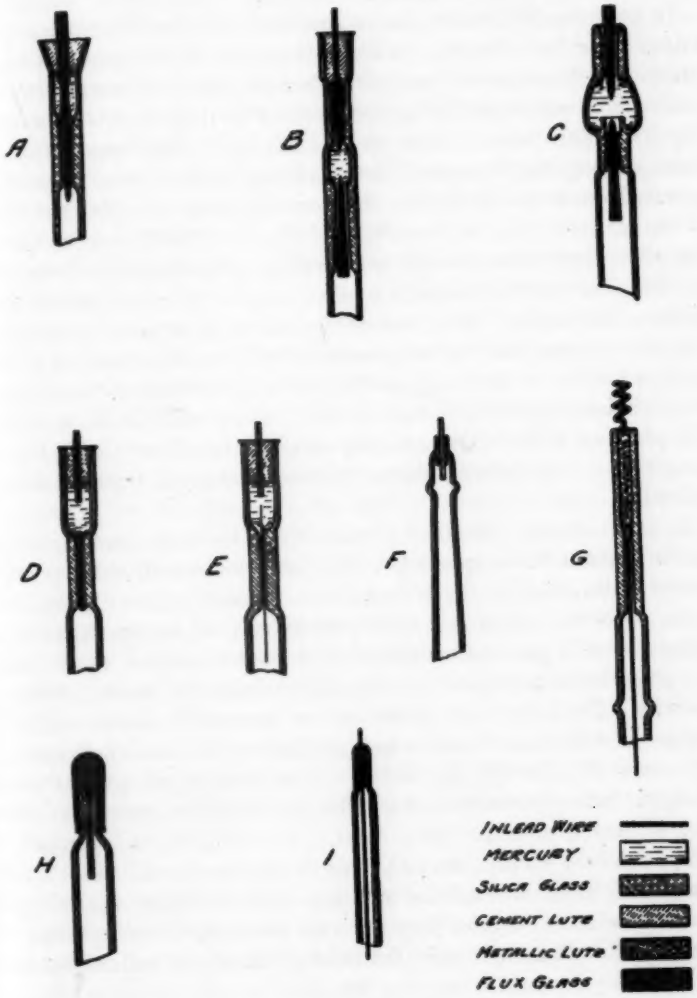

FIG. 1.—(a) *Earliest invar in-lead joint*; (b) *Later invar joint*; (c) *Modern invar joint*; (d) *Bastian type*; (e) *Kent-Lacell*; (f) *Kraus tube joint*; (g) *Sand's type*; (h) *Dever's type*; (i) *Keyes-Kraus joint with tungsten in-lead.*

with mercury, and with those previously described it is more uniquely adapted to the quartz mercury arc lamp for which it was developed than to any other apparatus. This joint is successfully used today in quartz-mercury arcs of French manufacture.

In 1912 Kraus[11] devised and patented a joint, Fig. 1F, of the current inlead type but differing radically from the above in principle. A platinum tube of one to two millimeters diameter and a wall thickness of about .05 mm is sealed by the aid of a flux to the quartz as shown in Fig. 1F. This joint is made up at a relatively high temperature and upon cooling the platinum tube is placed under tension through its greater contraction relative to the silica. In some cases the elastic limit of the platinum may be exceeded and the tube enlarged, in which case the silica may exert a slight compression upon the tube when again heated. The quartz is shaped for maximum strength to withstand these rather great strains. This joint has been used in commercial apparatus and many quartz mercury arcs made in 1912 are still operative with no signs of leakage at the inlead joints. Many laboratories have developed the technique of this joint and use it in the construction of apparatus. The platinum joint can theoretically be made directly to quartz, but the nearness of their melting points, Table 1, makes it a rather difficult operation.

In 1914, Sands[12] described a joint, Fig. 1G, to be used to seal in lead-in wires to silica apparatus. The joint has since been used extensively in England in the manufacture of silica mercury arcs, radio bulbs, spectrum tubes, etc. The process may be briefly described as follows: a short piece of molybdenum or tungsten wire is fitted loosely in a silica tube constricted to a capillary about the middle portion of the wire. The lower end of the tube is temporarily closed while the upper end of the tube connects by a capillary with a small bulb containing a carefully cleaned piece of lead. The whole is evacuated through the upper bulb to a fraction of a millimeter, while the lower part of the tube is heated and the silica constriction softened and pinched or collapsed by the air pressure to a tight fit on the wire. The lead is then heated and filters through the capillary into the highly heated space about the wire where it projects from the silica constriction. While the lead is still molten air is admitted through the upper bulb and presses the lead into the constriction to a firm seat between the silica tube and the wire. The more detailed description of the technique from which

[11] Kraus, U. S. Pat. 1,093,997; 1914.
[12] Sand, Proc. Phys. Soc. Lond. *26*, p. 127; 1914.

the above was abstracted will be found by reference to the patent disclosure of Sand and Reynolds.[13]

Devers[14] has recently disclosed an in-lead type of joint shown in cross section in Fig. 1H. The process consists essentially of casting about and in the end of a silica tube a plug and cap of metal such as copper the parts being so proportioned as to withstand the rather great pressure exerted upon the silica tube by the contracting metal. The joint has not as yet been used in commercial apparatus but would seem to be entirely practical. It has the same fundamental virtue as the other successful joints that when used in high temperature apparatus the internal strains are lessened rather than increased by the conditions of operation.

Hulbert[15] has described a method of sealing silica to glass by the use of silver chloride as a cement. Silver chloride freshly prepared by precipitation from an aqueous solution of silver nitrate with sodium chloride is washed, dried and ground to a powder. The end of tubes and the edges of plates, as the case may be, are ground to a preliminary fit, warmed to above 500° C. and dipped quickly into the silver chloride powder. The adhering powder is then melted evenly down around the edge of the glass. The parts are then brought together and heated until the fused silver chloride has run together much as does solder on properly tinned metal work. Hulbert even found it practical with care to seal quartz plates to glass tubing by this method. Following somewhat the technique of the Sand seal it is also possible to make an in-lead joint using fused silver chloride instead of lead.

For use as current in-leads in spectrum tubes and other very low current devices it is possible to fuse directly into silica molybdenum or tungsten wires of a diameter not exceeding .001″. A number of such wires may of course be used in multiple. The operation of fusing the silica onto the wires must be performed with the wire in a vacuum following in a general way the technique outlined in connection with the Sand seal.

All of the above described silica-to-metal joints are essentially limited to the sealing of silica to a wire in-lead although the invar pin and the platinum tube seals may be used to make connections to a metal system for the handling of small quantities of liquids or gases.

[13] Sand and Reynolds, U. S. Pat. 1,118,812; 1914.

[14] Devers, U. S. Pat. 1,531,265; 1925.

[15] Hulburt, "Sealing Quartz to Glass with Silver Chloride," Science, *56*, p. 147; 1922.

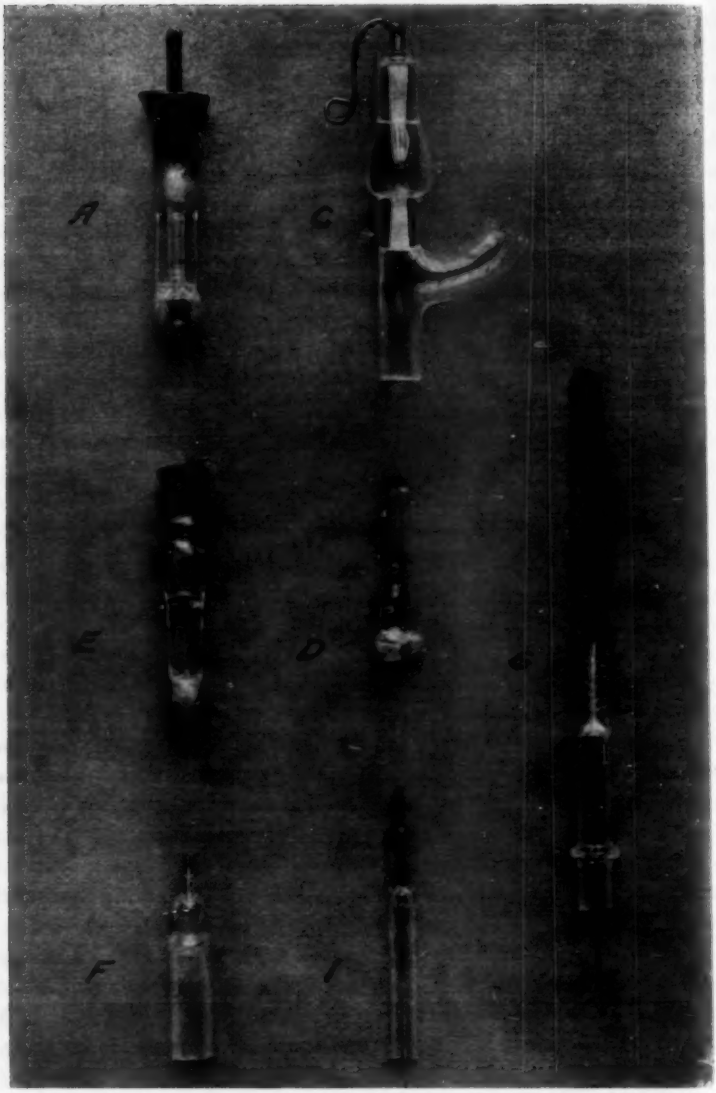

FIG. 2.—*In lead joints from commercial apparatus, designations correspond to Fig. 1.*

A type of silica-to-glass and thence to metal joint of much broader application than any of the above is that developed by Keyes and Kraus[16] Fig. 1I and used in conjunction with a glass-to-tungsten joint by Weintraub.[17] In this joint there is interposed between the silica and ordinary glass a series of intermediate glasses fused directly to the tungsten on the one hand and increasing successively in their coefficient of expansion to the point where a direct fusion to a common glass or to a metal is possible. A hard glass known by the trade name of Pyrex and the formula number of G702EJ and a closely rated lead glass known as G702P have come into common use in laboratory work as well as in the construction of commercial rectifier bulbs and high intensity incandescent lamps. As disclosed in the above patents it is possible to make up glasses intermediate between silica and the Pyrex glasses either by fusing together pulverized mixtures of the two or by modifying the composition of either glass by the addition of metallic oxides. The actual production of such glasses having suitable expansion coefficients, stability and workability is difficult and the manipulative skill required in their use is such as is not ordinarily acquired except by professional glass workers. For joining the Pyrex glasses to common lime or lead glasses certain glass companies are able to supply the necessary intermediates in the smaller sizes, and such joints have been made in many laboratories. While each of the several intermediate glasses used in this construction differ distinctly in composition, melting points, and coefficient of expansion from the other glasses adjacent to it in the joint the glasses must be so completely fused with each other that each very largely loses its identity and there results a final glass composition in the graded joint whose properties vary gradually and continuously over the required range. In this connection it should be noted that Houskeeper has described a method of making this same joint from Pyrex glass to lead glass with a thin walled copper tube.

Figs. 2 and 3 suggest some of the possibilities in graded joints of this type and their combinations with the glass to metal joints described by Kraus[1] and Houskeeper,[6] offer a solution of practically all laboratory problems. While information is unfortunately not available as to the composition and preparation of these intermediate glasses experience in their fabrication in sizes up to 2″ in diameter has fixed the following relationships which are largely independent of the particular glasses used. The length of the graded portion of a joint from silica to G702P

[16] Keyes and Kraus, U. S. Pat. 1,014,757; 1912.
[17] Weintraub, U. S. Pat. 1,154,081; 1915.

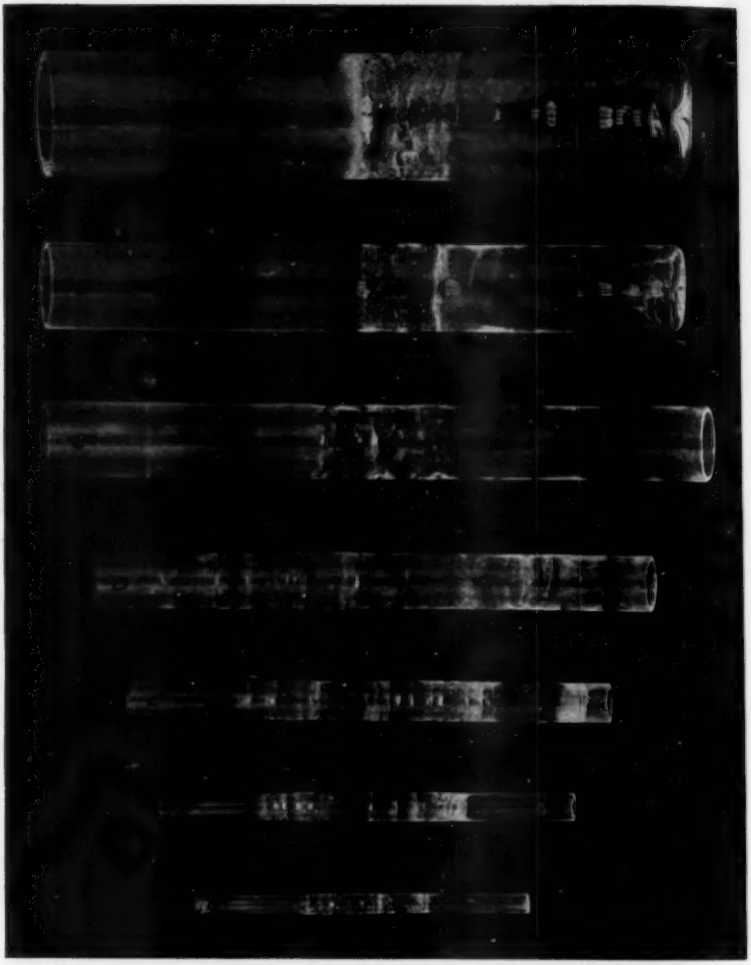

FIG. 3.—*Keyes-Kraus graded joints. Unretouched photograph showing distribution of flux glasses.* (a) *and* (b) G702P-*to-silica with end of silica tubing closed,—graded portion short as possible, bore 1-1/2 in. and 1 in.*; (c) G702P-*to-silica, bore 7/8 in.*; (d) (e) (f) (g) *Lead and lime glass-to-silica joints, bore 5/8 in. to 1/16 in.*

must be double its diameter, from G702P to lead or lime glass three times its diameter and from silica to lime or lead five times its diameter. In general, no attempt is made to work the material at either end of such a joint closer to the joint itself than its diameter, that is to say a length of silica, G702P or lead glass equal at least to the tube diameter is left at each end of the joint for the subsequent sealing on of additional tube stock or apparatus. Wall thicknesses of 1/32″ to 3/64″ are most desirable while those in excess of 1/16″ have generally been found impractical. It is difficult to define any arbitrary test of such joints, but as a minimum requirement they should withstand as great temperature, temperature changes and pressure as will the lowest melting glass in the joint, G702P, lead or lime, as the case may be.

These joints are useful as a means of introducing through the walls of a silica apparatus tungsten in-lead wires of capacities up to 100 amperes in a single lead and Houskeeper in-leads carrying thousands of amperes. They can be used as a means of fusing ultraviolet transmitting silica-glass windows to an apparatus of ordinary glass, as recently reported by Stockbarger.[18] They make possible the use in the same glass apparatus of three or more types of glass materials as, for example, a Pyrex mercury diffusion pump evacuating a quartz glass combustion tube through a gas handling system of ordinary glass. One such use has been recently reported by Schumacher[19] in which an apparatus consisting of bulbs of soda-lime glass, Pyrex G702P, Pyrex G702EJ, and silica-glass were all connected closely together in series and maintained at as greatly different temperatures as the softening temperatures of the various glasses would permit.

ENGINEERING DEPARTMENT,
COOPER HEWITT ELECTRIC CO.,
HOBOKEN, N. J.
July 18, 1925.

[18] Stockbarger, J.O.S.A. & R.S.I. *9*, p. 337; 1924.

[19] Schumacher, "The Wetting of Glasses by Mercury," J. Am. Chem. Soc. *45*, p. 2255; 1923

Incitation of the Reaction between H_2 and Cl_2 by Light.—The reaction between these two gases, by which HCl is formed, is a well-known instance of a reaction proceeding only when the mixture is illuminated; presumably because the Cl_2 molecules are put into an "excited state" by the light, as the other possibilities seem to have been pretty thoroughly eliminated by test experiments. In the present experiments Taylor measures the rate of reaction produced by eight ranges of the visible spectrum isolated by means of eight color filters, and calibrated as to energy-intensity by means of thermopile measurements Great precautions were required to prevent the rates of reaction from being distorted by mysterious "inhibiting agents." Light of wave length superior to 5000A produced no effect; but the rays transmitted by blue-green, blue and violet filters produced progressively greater effects per unit energy intensity. This suggests that the action upon the Cl_2 molecule is correlated with the continuous (unresolved) band in the Cl_2 absorption-spectrum, not with the resolved bands—[W. Taylor, Newcastle-on-Tyne; Phil. Mag. *49*, p. 1165-1168; 1925.]

KARL K. DARROW

Quantum Absorption in a Photographic Process.—Having previously found (cf. This Journal, *9*, pp. 321-322) that light dissociates silver bromide dispersed in gelatine at the rate of one silver atom liberated per quantum absorbed, Eggert and Noddack apply the same method to silver chloride dispersed in gelatine, with similar results. Light of wave length 365 mμ liberates, according to successive measurements, 81 to 87 per cent as many silver atoms as there are quanta absorbed; this the authors regard as a "sufficient proof" of the quantum law. The method and the (rather large) corrections required are for chemists to judge. In principle the method might furnish an additional experimental value of h.—[J. Eggert, W. Noddack; ZS. f. Phys. *31*, pp. 922-941, 1925.]

KARL K. DARROW

MEASUREMENTS WITH A LUMMER GEHRCKE PLATE

By John K. Robertson

In a previous article[1] attention has been directed to the desirability of supplementing formal analytical lectures in physical optics by suitable quantitative laboratory experiments. While a glance at such a book as *Taylor's College Manual of Optics* shows how readily this can be done, some of the more modern instruments do not seem to have received the attention they deserve. It is surprising, for example, to find in such an excellent book no reference to the Lummer Gehrcke plate. In this paper, therefore, a brief account is given of a series of measurements which can be made conveniently with that instrument by any senior student. The measurements given were made visually by two of the writer's students, Mr. A. L. Kuehner and Mr. C. Y. Hopkins.

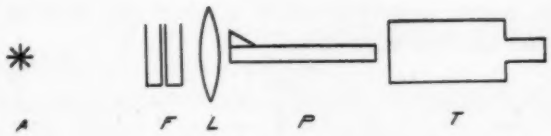

Fig. 1.—*Arrangement of apparatus for observation of fringes.*

The experimental arrangement is extremely simple. Light from a mercury arc lamp A, filtered (by solutions of potassium dichromate and neodymium chloride) so as to transmit λ 5461 is condensed by the lens L, so that an approximately parallel beam falls on the Lummer plate P. The plate, with reflecting faces vertical, rests in a horizontal position on the table of a spectrometer (whose collimator has been put to one side), while the interference system is observed in the telescope T previously focussed for infinity. With such an arrangement literally hundreds of vertical fringes may be seen as the telescope is rotated to receive light reflected at different angles.

To make clear the use to which the measurements may be put a glance at the simple theory of the plate is necessary.

If t = thickness of the plate,
p_n = the order of interference,
i_n = the angle of emergence for the n^{th} fringe from the centre,
μ = index of refraction for wave length λ,

[1] Robertson, J.O.S.A. *9*, 5, p. 611; 1924.

then

$$p_n = \frac{2t}{\lambda}\sqrt{\mu^2 - \sin^2 i_n} \tag{1}$$

or

$$= \frac{2t}{\lambda}\sqrt{\mu^2 - \cos^2 \phi_n} \tag{2}$$

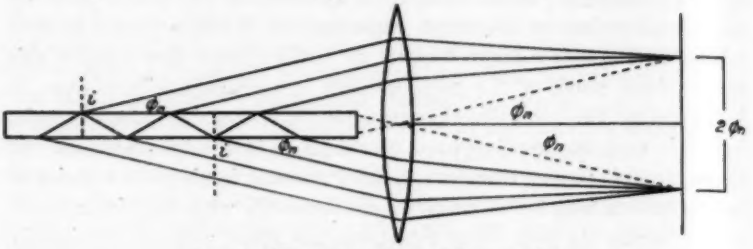

FIG. 2.—*Illuminating theory of Lummer plate.*

where ϕ_n = one-half the angular separation of the n^{th} fringes on either side of the centre, numerical values of which are given in Table 1, column 2, for ten successive orders.

TABLE 1.

n	ϕ (degrees)	ϕ (radians)	$\phi_{n+1}^2 - \phi_n^2$
10	2° 1′ 45″	.0354	1.37×10^{-4}
9	1° 54′ 55″	.0334	1.36 "
8	1° 47′ 45″	.0313	1.27 "
7	1° 40′ 10″	.0292	1.44 "
6	1° 31′ 25″	.0266	1.51 "
5	1° 21′ 15″	.0236	1.47 "
4	1° 10′ 15″	.02025	1.32 "
3	57′ 20″	.01670	1.39 "
2	40′ 25″	.01178	1.36 "
1	16′ 30″	.00480	
		Average	1.388×10^{-4}

CALCULATIONS

1. *Constancy of* $\phi_{n+1}^2 - \phi_n^2$. Since ϕ_n is a small angle it may readily be shown[2] that

$$\phi_{n+1}^2 - \phi_n^2 = \frac{\lambda\sqrt{\mu^2 - 1}}{t}. \tag{3}$$

It follows, therefore, that for any given λ, $\phi_{n+1}^2 - \phi_n^2$ should remain constant. That this is the case will be seen from the numbers in column

[2] McLennan, Proc. Roy. Soc. *90*, p. 244; 1914.

4, Table 1. While the order of accuracy is not high, it must be remembered that angles of the order of only a degree or two have been measured visually with an ordinary spectrometer reading to 20″. For greater accuracy the fringes may be photographed, but this necessitates much more time and is not always desirable in laboratory courses.

II. *Thickness of the Plate.*—By using the mean of all values of $\phi_{n+N}^2-\phi_n^2$, formula 3 may also be used to calculate t, the thickness of the plate, a constant supplied by the manufacturer. To do so it is necessary to know the value of μ for the particular wave length used in the experimental observations. By plotting a dispersion curve from the indices for standard wave lengths supplied by the manufacturer, or by obtaining the constants of a Cauchy dispersion formula from the same data, the value of μ for any λ is readily obtained. For λ 5461 the students' dispersion curve gives $\mu=1.5192$. It follows that

$$t=\frac{\sqrt{(1.5192)^2-1}\times 5461\times 10^{-8}}{1.388\times 10^{-4}}$$

$=4.500$ mm, a value 0.04% higher than 4.498 mm, that supplied by Adam Hilger and Co.

III. *Order of Interference.*—(a) From the fundamental equation, the value of p for grazing incidence, that is, for $i_n=90°$, may at once be obtained. For $\lambda=5461$, we find, from equation 1,

$$p_n=\frac{2\times 0.4498\times\sqrt{(1.5192)^2-1}}{5461\times 10^{-8}}$$

$$=18.85\times 10^3 .$$

(b) By making use of the angular separation of any two orders, p_n may be found in another way, which is independent of the value of the index.

If $p_{n+N}=$ order of the $(N+n)$th fringe, then, since

$$p_{n+N}=\frac{2t}{\lambda}\sqrt{\mu^2-\cos^2\phi_{n+N}}\,, \tag{4}$$

it may easily be proved, combining (2) with (4), that

$$p_n=\frac{2t^2}{N\lambda^2}(\phi_{n+N}^2-\phi_n^2)-\frac{N}{2} \tag{5}$$

$$=\frac{2t^2}{N\lambda^2}(\phi_{n+N}^2-\phi_n^2)\,, \tag{6}$$

to a high order of accuracy.

If we take $N=1$, then we may use for $\phi_{n+N}^2-\phi_n^2$ the average of the values given in the last column of Table 1 and we have

$$p_n=\frac{2\times.4498^2\times1.388\times10^{-4}}{(.00005461)^2}$$

$$=18.83\times10^3\,.$$

IV. *Calculation of the Index.*—From equation (3) we have at once

$$\sqrt{\mu^2-1}=\frac{t}{\lambda}(\phi_{n+1}^2-\phi_n^2)$$

$$=\frac{.4498\times1.388\times10^{-4}}{5461\times10^{-8}}\,,$$

from which

$$\mu=1.519\,.$$

V. *Value of $\Delta\lambda_m$ the Change in Wave Length which Corresponds to a Shift of one Fringe.*—If ϕ_n' = angle corresponding to the nth fringe for a wave length $\lambda+\Delta\lambda$, it is not difficult to prove that

$$\Delta\lambda=\frac{\phi_n'^2-\phi_n^2}{\phi_{n+1}^2-\phi_n^2}\cdot\frac{p_n\lambda^2}{p_n^2\lambda-4t^2\mu\dfrac{\partial\mu}{\partial\lambda}} \tag{9}$$

$$=\frac{\phi_n'^2-\phi_n^2}{\phi_{n+1}^2-\phi_n^2}\cdot\Delta\lambda m \tag{10}$$

where

$$\Delta\lambda m=\frac{p_n\lambda^2}{p_n^2\lambda-4t^2\mu\dfrac{\partial\mu}{\partial\lambda}}\,, \tag{11}$$

is evidently the value of $\Delta\lambda$ for which the pth order for wave length $\lambda+\Delta\lambda$ just falls on the $(p+1)$th order for wave length λ. To evaluate $\Delta\lambda_m$ it is necessary to know the magnitude of $\frac{\partial\mu}{\partial\lambda}$. For any λ this may be obtained either from the slope of the dispersion curve, or from Cauchy's dispersion formula. For λ 5461 the value, as obtained from the students' report by the second method, is -788. We have, therefore,

$$\Delta\lambda m=\frac{18.84\times10^3\times(.00005461)^2}{18.84^2\times10^6\times0.00005461+4\times.4498^2\times1.5192\times788}$$

$$=0.305\times10^{-8}\text{ cm}$$

$$=0.305\,A.$$

Should a satellite be associated with a main line, the corresponding $\Delta\lambda$ is at once obtained from equation 10, which may now be written,

$$\Delta\lambda = \frac{\phi_{n'}^2 - \phi_n^2}{\phi_{n+1}^2 - \phi_n^2} \times 0.305\ A\ .$$

VI. *Resolving Power.*—The value of $\Delta\lambda_m$ gives an idea of the upper limit of the range of wave lengths over which this particular plate is useful. A knowledge of the lower limit is generally of greater importance, but the resolving power cannot readily be calculated with any high degree of accuracy. Following Simeon,[3] its value may be estimated for grazing incidence by means of the relation

$$\text{Resolving Power} = \frac{l}{\lambda}\left(\mu^2 - 1 - \lambda\mu\frac{\partial\mu}{\partial\lambda}\right). \qquad (12)$$

For $l = 13$ cm and $\lambda = 5461$ A. this gives a magnitude 327×10^3, according to which the smallest value $\Delta\lambda$ could have in the neighborhood of this wave length would be 0.017 A.

As Simeon points out, however, this relation is derived using the total aperture of the beam entering the telescope, on the assumption that all the interfering rays are of equal intensity. While he calls attention to the incorrectness of this assumption, particularly when grazing incidence is not the case, Simeon states that values obtained in this way agree closely with those found in actual practice.

To calculate accurately the resolving power use must be made of (1) the intensity distribution in a single order as given by Lummer and Gehrcke,[4] (2) the generally accepted assumption of the late Lord Rayleigh that the limit of resolution is reached when two wave lengths are separated such a distance that the intensity one-half way between their centres is equal to 0.8 of that at either center. As the expression for intensity distribution involves the coefficient of reflection, a quantity which depends on the angle of incidence, and as the procedure is very laborious, little is gained by the more elaborate method. An estimate made by the writer seems to indicate, however, that Simeon's formula gives values appreciably higher.

As a matter of fact a knowledge of the exact value of the resolving power is never necessary. In actual practice one is concerned with such things as wave lengths of satellites, and for such purposes, formula 10 is all that is necessary.

QUEEN'S UNIVERSITY,
KINGSTON, CANADA,
JUNE, 1925.

[3] Simeon, Jour. of Sci. Inst. *1*, (10), p. 296; 1924.
[4] Lummer and Gehrcke, Ann. der Phys. Band *10*, p. 457; 1903.

Relative Intensities of Lines Belonging to Multiplets.—After an account of refinements in experimental technique enabling him to measure relative intensities of lines in the ultraviolet and not too far apart, Dorgelo cites data for some lines in the spectrum of ionized calcium. The intensities of the two lines of the first doublet in the sharp subordinate series stand very accurately in the simple ratio 2 : 1, but for a doublet in the principal series the observed values of the ratio depart very much from 1 : 2, of which the author naively says "I do not doubt that the ratio would be 1 : 2 if self-reversal could be entirely eliminated." Measurements upon a *pd* triplet give satisfactory ratios.—[H. B. Dorgelo, Eindhoven; ZS. f. Phys. *31*, pp. 827-835; 1925.]

KARL K. DARROW

Arrangement of Atoms in Mother-of-Pearl.—X-ray analysis of samples of mother-of-pearl, carried out by a method which is a sort of cross between Laue's and Bragg's, reveals that the substance consists largely of crystals of aragonite, all lying with a particular axis normal to the plane of the laminations in the substance, but otherwise oriented at random. The other materials involved in the mother-of-pearl are evidently supposed to form capsules enclosing the crystals.—J. H. Shaxby, laboratory of M. de Broglie; Phil. Mag. *49*, pp. 1201-1206; 1925.]

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